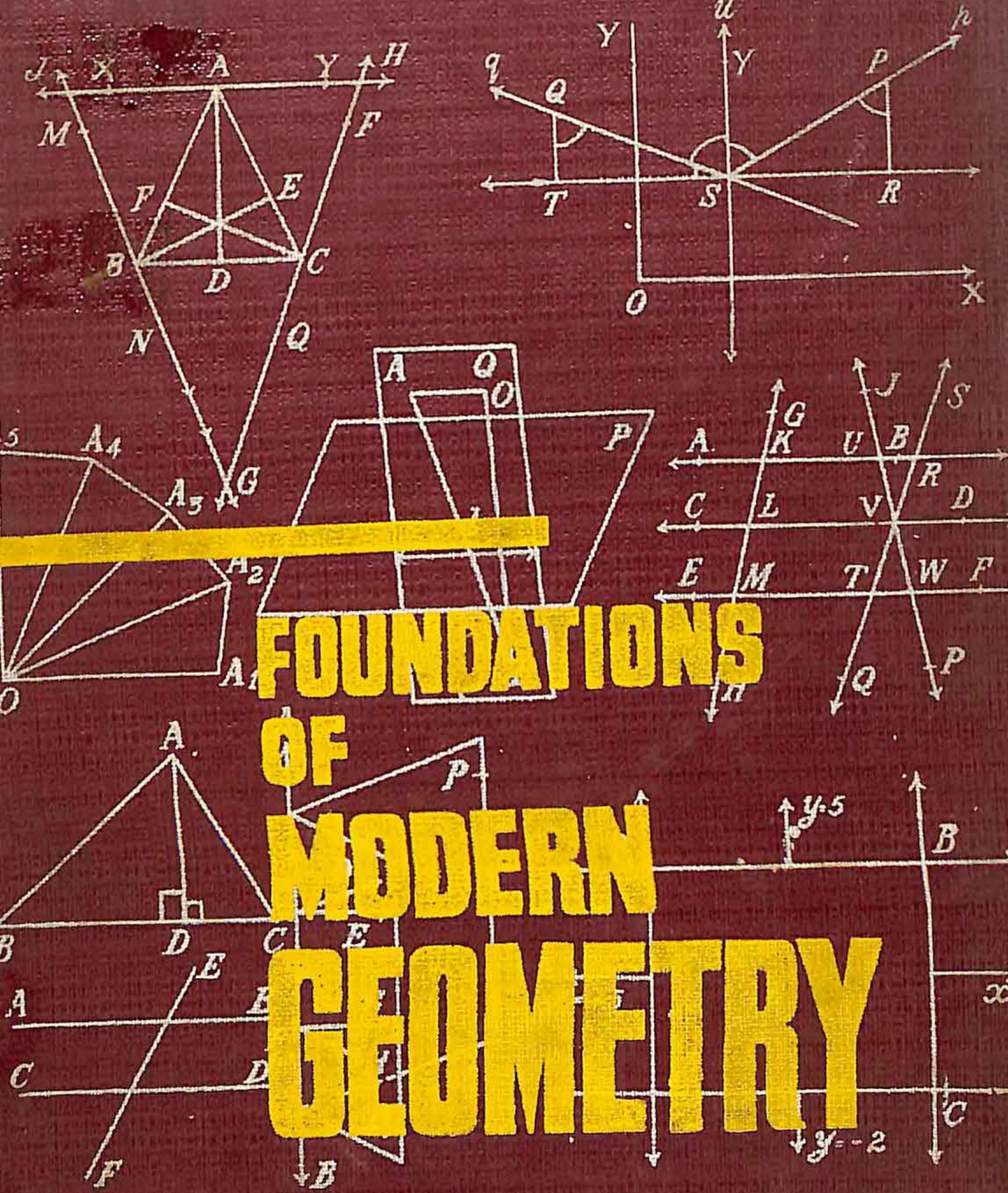
S. S. SACHDEVA



FOUNDATIONS OF MODERN GEOMETRY

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FOUNDATIONS
OF
MODERN GEOMETRY

by

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PREFACE

> "Mathematics is that subject in which we never know what we are talking about, nor whether what we are saying is true"
>
> —*Bertrand Russell*

This book has been primarily written with a view to acquaint the classroom teacher with the latest trends that are being incorporated in the presentation of a school course in geometry in school systems throughout the world. However this can also prove useful to a bright school student or to one who intends to prepare for science talent search competition or other such competitions or to one who is preparing for adopting teaching as a profession.

While writing this book the aims that have been kept in view are :

(*a*) To present geometry as an axiomatic system as the axiomatic method is the basis of almost all the new Maths that is being developed in the world.

(*b*) To enable the reader know that hundreds of geometries can be developed.

(*c*) To remind the classroom teacher that the applicability of the Euclidean geometry that he has all along been teaching in the classroom is limited to the sheet of the paper or to the very immediate neighbourhood of the space in which he lives.

(*d*) To acquaint the reader with the deficiencies that exist in the system as developed by Euclid in his elements and to suggest methods of removing them.

As very aptly portrayed by Bertrand Russell's quotation given above, Maths is essentially abstract in nature and efforts have been made to maintain this spirit throughout this book.

In the very first chapter of the book, Axiomatic method has been explained and the reader has been introduced to a little of elementary logic and the nature of proof. Euclidean geometry has been developed as an axiomatic system in chapters III to XV and in these efforts have also been made to correlate analytical methods and synthetic methods. In chapter XVI, the reader has been introduced to the concept of finite geometries and various other non-Euclidean geometries.

It was in a Summer Institute held at Darjeeling under the auspices of North Bengal University in the summer of 1963, that I was introduced to Axiomatic method. Subsequently I got a little deeper understanding of this when I taught Maths at Central High School Springfield, Missouri, U.S.A. for a school term and during the courses I took up at Missouri University at Rolla and Utah State University, Logan, Utah, U.S.A. I express my sincere gratitude to all of my teachers from whom I had the privelege to learn at these Institutes.

It was during my tenure as lecturer at State Institute of Education, Delhi that the academic environment prevalent there inspired me to write this book and Shri B.D. Bhatt, our worthy Director of Education, Delhi was kind enough to permit me to do so. I express my sincere thanks and gratitude to the Director of Education for this.

I am highly thankful to Shri C.S. Sarna who is a genious personality and is an author of a number of books on Mathematics. He helped me a lot in preparing this book. I am also thankful to S/Shri B.S. Ahuja and Satnam Singh for going through portions on Analytical geometry. I need also to mention the assistance and co-operation extended to me by my wife and children during the one and half years of hard labour put in by me in writing the manuscript and revising it again and again. I will gratefully acknowledge any improvements, mistakes, errors etc. suggested by my friends in Maths.

I also express my sincere gratitude to the publishers for taking so much pains in bringing this book out.

New Delhi,
July 28, *1968.*

S. S. SACHDEVA

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LIST OF SYMBOLS USED

$a \in A$:—a is an element of set A.
$\{\ \}$:— Notation for a set.
$\{a\}$:—The set of which a is the only element.
ϕ or $\{\ \}$.—Empty set or null set.
$\{3, 4, 9\}$:—The set whose elements are 3, 4 and 9.
$R = \{x \mid x \text{ is a real}\}$:—The set of all real numbers.
$\{x \mid x \in A \text{ or } x \in B\}$:—x belongs to A or to B.
(a, b) :—An ordered pair whose first element is a and the second element is b.
$A \times B$:—The set of all ordered pairs (a, b) where a belongs to A and b belongs to B.
U :—The Universal set.
V :—Venn diagram.
$A \subseteq B$:—Set inclusion ; A is subset of B or B includes A.
$A \subset B$:—Proper set inclusion or A is proper subset of B.
$A \supseteq B$:—A contains B or B is a subset of A.
$A \supset B$:—B is proper subset of A.
$A \cap B$:—Intersection of A and B.
$A \cup B$:—Union of A and B.
A' :— Complement of A.
p, q, r :—Propositions.
p :—Proposition p is true.
$\sim p$:—Negation of p or p is not true.
$p \wedge q$:—Conjunction of p and q or propositions p and q.
$p \vee q$:—Inclusive disjunction of p and q.
$p \rightarrow q$.—If p then q or p implies q.
$p \leftrightarrow q$:—p is equivalent to q.
$|a|$:—Absolute value of a.
$>$:—Greater than.
$\ngtr$:—Not greater than.
$\geqslant$:—Greater than or equal to.
$<$:—Less than.
$\nless$:—Not less than.
$\leqslant$:—Less than or equal to.
$\exists$:—There exists.
$\nexists$:—There does not exist.
$\forall$:—For all.
$\parallel$:—parallel to.

$\nparallel$:—Not parallel to.

$\perp$:—Perpendicular to.

$\cong$:—Congruent to.

$\sim$:—Similar to.

$\overleftrightarrow{AB}$:—Line containing the points A and B or Line AB.

$\overline{AB}$:—Segment whose end points are A and B or segment AB.

$\overrightarrow{AB}$:—Ray AB.

$\overset{\circ\!\!\rightarrow}{AB}$:—Half line AB.

$m(\overline{AB})$ or $m\overline{AB}$ or d (A, B) or AB :—Measure of line segment AB.

$\angle ABC$:—Angle ABC.

$m\angle ABC$:—Measure of angle ABC.

$\triangle ABC$:—The triangle whose vertices are A, B and C or the triangle ABC.

Isos. '—Isosceles.

quad. :—Quadrilateral.

$\|^{gm}$:—Parallelogram.

rect. :—Rectangle.

$\odot$:—Circle.

def. :—Definition.

no. :—Number.

$=$:—Equal to.

$\neq$:—Not equal to.

$\in$:—Belongs to.

$\notin$:—Does not belong to.

$a \mathrm{R} b$:—a has a relation with b.

T :—True.

F :—False.

$\{P_1,P_2\}$:—Line determined by points P_1 and P_2.

$\overleftrightarrow{P_1P_2P_3}$ or $\{P_1, P_2, P_3\}$:—Plane determined by three non-collinear points P_1, P_2, P_3.

A—B—C :—Three collinear points A, B, C such that B is between A and C.

1—1 :—One-to-one correspondence.

$P(a,b)$:—A point P whose x-co-ordinate is a and whose y-co-ordinate is b.

CHAPTER I

THE AXIOMATIC METHOD

Growth of Geometry—All one's life one learns about geometry ! One learns about it informally when one observes the shapes and sizes of things around him. As a student one learns something about geometry in the class-room. As a pupil one learns about the properties of figures composed of points and lines like triangles and quadrilaterals and later on at the high school stage, a pupil is exposed to propositions as compiled by Euclid in his books I, II, III and IV. However, he is rarely exposed to the study of geometry as a discipline—as an axiomatic system. Geometry as it has been studied in schools of European countries till recently and as it is being studied in schools in our country can hardly develop the qualities of logical thinking and reasoning in the pupils.

The word geometry has been derived from the Greek words '*ge*' and '*metrein*' which mean 'the earth' and 'to measure' respectively. The science of geometry originated in Egypt about three to five thousand years ago mainly as an art of earth measurement. Each year the river Nile overflowed and reinvigorated land in its neighbourhood but while so doing it wiped out the boundaries between adjoining fields of different landowners and these had to be re-mapped after each such flooding. Consequently, the Egyptians developed a remarkable body of techniques and rules for calculating simple areas and volumes and for drawing simple constructions mainly for the purpose of surveying lands and constructing canals, tombs and temples. The geometry developed by Egyptians mainly consisted of statements of results. It hardly contained any general rules of procedure.

The earliest mention of some problems in geometry has been found to be contained in 'Papyrus paper' which is said to have been written by Ahmes some time before 1700 B.C. and which is now preserved in British Museum. This document and which is of Egyptian origin, is said to have been copied from yet another document which is now lost and which is also of Egyptian origin and which is said to have been written about 3400 B.C.

Ancient Babylonians have also made very significant contributions to geometry and some Historians of Maths. contend that geometry originated in Babylon rather than in Egypt. It can, of course, be safely said that [illegible] originated in either of these two countries. It appears that both the Egyptians and Babylonians cared little for theoretical results. They only collected some techniques, codified them and passed them on to the succeeding generations for their day-to-day practical use. Of course it was a creditable achievement.

With the passage of time, this very useful body of knowledge reached the other nations. The Greeks in particular got very much interested in this art of land measurement and gave it the name 'ge-metrein'. Greeks were temperamentally inclined towards philosophy and abstraction and they had ample leisure and security. Geometry was an art till it reached the Greeks as then it was merely a disorganised collection of rules for computing simple areas and volumes, some of which were correct and some incorrect and had been developed mostly on the basis of inductive reasoning. The Greeks took it up and recast it in the form of a deductive science. The Greeks possessed remarkable ingenuity. They could visualise that there lay hidden within the Ægyptian techniques of measurement of land some general principles which applied not only to land and its measurement but also to relationships among spatial figures. The Greeks thought of abstract surfaces and volumes bounded by abstract lines while the thinking of Egyptians was only limited to plots of land and some physical devices to demarcate their boundaries. The Greeks proceeded to develop geometry without any reference to 'land' and its measurement. The Greeks dealt with geometrical figures like triangles which could only be imagined but it had enormous practical consequences as it helped them to create a model of the manner in which the stars and planets behaved. They developed Mechanics and Physics with its help.

Thales (640—546 B.C.), a Greek merchant, studied for some years in Egypt and during his stay there he became interested in geometry. He is said to be the first to apply theoretical geometry to practical problems. The invention of theorems on (*i*) the equality of base angles of an isosceles triangle, (*ii*) the congruence of two triangles with A.S.A. congruency, (*iii*) the bisection of a circle by any of its diameters is said to be due to him. Thales taught geometry to his friends and one of his pupils was Pythagoras (born about 580 B.C.). Pythagoras and Pythagorean school consisting of his pupils developed many theorems and this process continued till 300 B.C. when Euclid, an Egyptian Jew and a teacher of Mathematics at Alexendria collected these theorems, gave them some sort of a logical order and wrote them in his famous 'Elements'. He divided the 'Elements' into thirteen chapters called books. Although these were written some 2300 years ago, still these are considered by far to be the best contribution to the whole body of Mathematics. Till the beginning of this century, the 'Elements' was prescribed as a text book in the schools in some of the European countries, particularly England.

Euclid started from a few simple assumptions, axioms and postulates, some definitions and built up a well organised chain of theorems with the help of laws of logic. This logical deductive pattern which is abstract in nature is known as 'Axiomatic system'.

Importance of Euclid's contribution can be realised from the fact that even today most of Mathematics is constructed on the same pattern as was initiated by Euclid. The world has so far failed to produce any text book whose contribution or impact on the human knowledge can come anywhere near Euclid's 'Elements'.

Euclid's 'Elements' reigned undisputedly over the Mathematical world till 15th or 16th century when doubts began to be expressed about some of the postulates and definitions. However, before we discuss these it would be in the fitness of things to know what is an axiomatic system and what were the axioms and postulates formulated by Euclid.

What is an Axiomatic System ?—An axiomatic system consists of the following :—

(1) A set of undefined terms which forms the basic building blocks of the system.

(2) A set of initial assumptions—Axioms and/or postulates which are accepted without any proof.

(3) The laws of Logic.

(4) A set of theorems which are derived from Axioms and/or postulates with the help of laws of Logic.

(5) 'Riders' to the theorems or applications of theorems.

Every Mathematical Science or an axiomatic system begins with a relatively small number of terms which are left undefined. In every system, some building blocks are needed and it is not possible to define every term, although we can interpret them intuitively or explain them with the help of 'physical models'. In every field, we need to start with some simpler terms which are taken to be understood. In case we try to define every term we shall simply be going in circles and reach the same term which we tried to define. Supposing you are looking up the dictionary for the meaning of a particular word 'A'. You will find that the meaning of word 'A' is given in terms of some words 'B' and 'C'. If you don't understand the meanings of word 'B' as well as 'C', then you will again look up the dictionary for the meanings of the words 'B' and 'C'. You may find yourself in the situation in which the meaning of 'B' is given in terms of 'C' and 'D' and meaning of 'C' might have been given in terms of

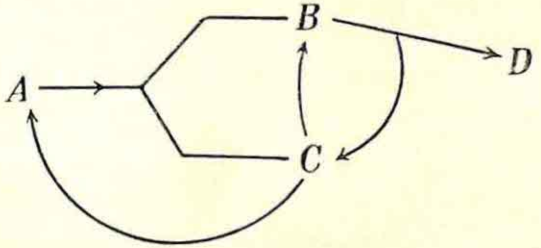

Fig. 1.

'B' and 'A'. Now if you do not understand the meanings of the word 'B', 'C' or 'D', then you come back to 'A'. We can represent this situation as given in Fig. 1.

An example of 'going in circles' would be as follows :—

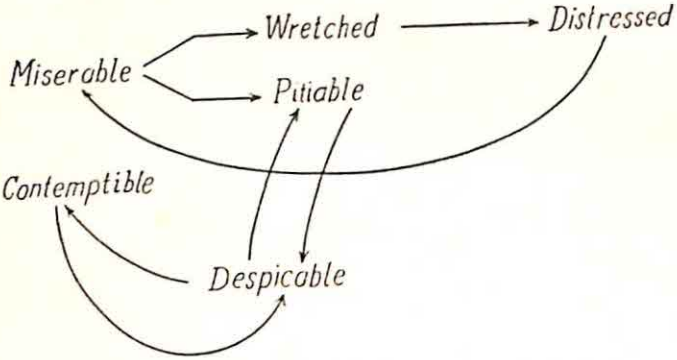

Fig. 2.

So you have to 'intuitively' understand the meaning of any one of miserable, wretched, pitiable, distressed, despicable, contemptible. Same is the case with any Axiomatic system. Of course the number of undefined terms is kept to the minimum.

Also it is not possible to prove every statement. Some statements are accepted without any proof. These statements are called initial assumptions or axioms or postulates. Everybody has to make assumptions in life. Supposing you want to travel to a city about a few hundred miles away on a particular day and you get your air passage booked. On the next day you read a news item in the newspapers that air flights are likely to be suspended on that day on account of threatened strike by air crew. Now supposing you have to visit that city on that day, then you will assume that air-flights are suspended on that day and then you will chalk out an alternative course of action :—

(1) You may think of booking a seat in the fast train.

(2) You may decide to borrow friend's car and travel in it, etc.

It may so happen that air flights may not be suspended at all.

An army general has to make many assumptions and chalk out a possible course of action.

'If the enemy makes an air attack at such and such time from such and such direction then he will take such and such defensive and/or offensive measures.' So the army general may have to have chalked out many a course of action out of which he may make use of one which he may consider most appropriate in a particular situation.

Scientists have been making initial assumptions and on its basis they have been developing systems and such systems have been found to have useful applications in the physical world. Numerous such examples can be cited. For instance, Newton was very much troubled by the problem of distinguishing 'relative motion' from 'absolute motion', so he postulated or made an assumption that 'in the remote regions of the fixed stars or perhaps far beyond them there may be some body at rest'; but he admitted that there was no way of proving it. This view prevailed during 18th and 19th century. The development of wave theory of light led the scientists to assume that space was some kind of substance and the scientists evolved a hypothetical substance 'ether' which they assumed must pervade all space and matter. Two American physicists, A. A. Michelson and E. W. Morley designed and performed an experiment and drew conclusions which would not support the 'Ether theory'. The famous Mathematician and Physicist Albert Einstein was one who pondered over the results of Michelson-Morley experiment. He rejected the 'Ether theory', made certain assumptions and developed 'Relativity theory', and the results derived from these assumptions have been found to be valid whenever these have been put to test.

In a like manner, a mathematician also proceeds on the basis of some assumptions without caring to know whether his assumptions are valid (true) or false. In fact, in some situations it is not possible to check the validity of assumptions beyond any doubt. One thing is of course certain that if the assumptions that we make are false then the conclusions that we will draw from those assumptions will also be false. If a mathematician develops an axiomatic system and a scientist finds that the conclusions that the mathematician has drawn have applications in the physical world, then the set of postulates taken by the mathematician must necessarily have some validity.

It may not be out of place to mention that undefined terms are often suggested by the objects in the physical world and assumptions or axioms or postulates are generally derived from the properties of objects in physical world. Mostly these axioms are so simple and obvious that these are likely to be taken as 'self-evident truths' which is not a correct approach. In fact, 'obviousness' and 'self-evident truths' do not have any place in the axiomatic approach. In traditional mathematics an axiom is something which is a self-evident truth while in modern approach an axiom is a statement which is to be accepted without any proof. A modern mathematician does not bother whether his axioms or postulates are self evident truths or not. Theorems are derived from these postulates with the help of laws of logic. Occasionally, the results so derived are made use of by the technologist to harness the forces of nature for the benefit of mankind. In other words, we start from nature, accept some undefined terms and postulates, make some definitions, derive some results with the help of laws of logic and apply those results to harness the nature. We can represent this process by the figure on page 6.

Of course, a mathematician never bothers to know if his undefined terms and postulates have any relevance to the physical world or the results derived by him could be of any use in harnessing nature.

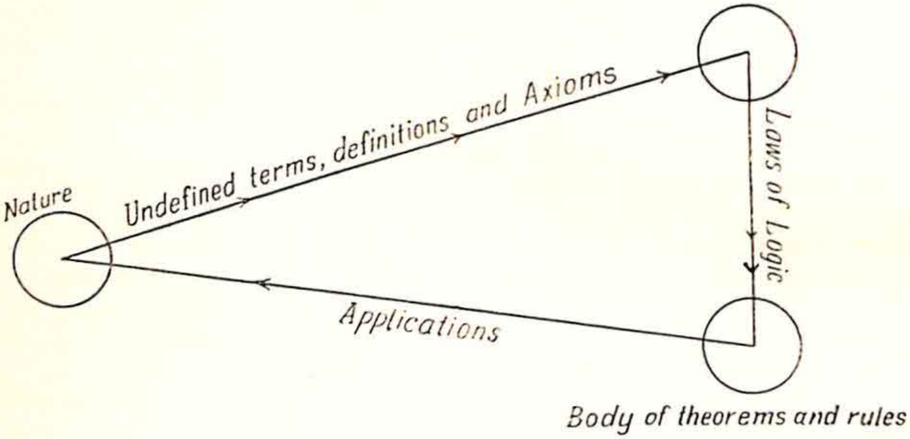
Nature
Undefined terms, definitions and Axioms
Laws of Logic
Applications
Body of theorems and rules

Fig. 3.

We have already suggested that if a mathematician wishes to establish a deductive system then whichever statements he makes, have to be derived from statements already proved by him and these in turn have to be derived from statements proved still earlier. As we cannot press back indefinitely without incurring the risk of 'circular reasoning', some simpler, primitive statements have to be assumed and these have to be accepted without any proof. We have been using the words, 'assumptions', 'axioms' or 'postulates' for these primitive statements. Now the 'assumptions' is a common word for these. 'Axioms' and 'postulates' is same thing as 'assumptions'. The Greeks regarded both the axioms and postulates as 'self-evident' truths—something undeniable and eternally verifiable. Those statements that were considered by them to be the general truths and which were also applicable outside the realm of geometry were called by them as Axioms but those statements which specifically concerned the geometric relations were called by them as 'postulates'. According to Aristotle, every demonstrative science must start from indemonstrable principles called Axioms.

As already stated, the choice of Axioms is not arbitrary in the purest of pure mathematics. The set of axioms has to be *consistent*. A set of axioms is said to be consistent if and only if there are no contradictions in the axioms and the theorems that can be derived from them. The consistency of an axiomatic system, therefore, depends on all the theorems, the discovered as well as undiscovered that can be proved in the system. Also the set of axioms has to be independent. The set of axioms will be said to be 'independent', if none of the axioms can be derived from or is a logical consequence of some other axioms. While selecting his set of axioms, a mathematician has not to bother about their 'self-evidence' or truth. He has only to see that those are consistent (or without any contradictions).

It were the Greeks who gave the world the guidelines for establishing a deductive system. A deductive system in the field of geometry is called Postulational or demonstrative geometry. The Greeks gave it the name of 'synthetic geometry'. As already stated, it was Euclid who compiled synthetic geometry. Therefore, it is also called Euclidean geometry. He doubtlessly drew upon his predecessors but the whole design of the system that he built up was his own. He built his system on five axioms, five postulates and twenty-three definitions and deduced 465 propositions in a logical chain. Of course, Euclid's axioms, postulates and definitions have been found to be inadequate and insufficient. Before we discuss their insufficiency and inadequacy, it would be appropriate to list those.

Axioms, Postulates and Definitions as given by Euclid in his 'Elements'.

Axioms : 1. The things which are equal to the same thing are also equal to one another.

2. If equals be added to the equals the wholes are equals.

3. If equals be subtracted from equals, the remainders are equals.

4. Things which coincide with one another are equal to one another.

5. The whole is greater than the part.

Postulates : 1. To draw a straight line from any point to any point (or, a straight line can be drawn from any point to any point).

2. To produce a finite straight line continuously in a straight line (or, a finite straight line can be produced continuously in a straight line).

3. To describe a circle with any centre and distance (or, a circle may be described with any centre and any distance).

4. All right angles are equal to one another.

5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely, meet on that side on which are angles less than two right angles.

Definitions : Euclid adopted about two dozen definitions which are as follows :—

1. *Point.* A point is that which has no part.

2. *Line.* A line is breadthless length.

3. The extremities of a line are points.

4. *Straight line.* A straight line is a line which lies evenly with the points on itself.

5. *Surface.* A surface is that which has length and breadth only.

6. The extremities of a surface are lines.

7. *Plane surface.* A plane surface is a surface which lies evenly with the straight line on itself.

8. *Angle.* A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

9. When the lines containing the angle are straight, the angle is called rectilineal.

10. *Perpendicular.* When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right and the straight line standing on the other is called a perpendicular to that on which it stands.

11. An obtuse angle is an angle greater than a right angle.

12. An acute angle is an angle less than a right angle.

13. A boundary is that which is an extremity of anything.

14. A figure is that which is contained by any boundary or boundaries.

15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.

16. And the point is called the centre of the circle.

17. A diameter of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle and such a straight line also bisects the circle.

18. A semicircle is the figure contained by the diameter and the circumference cut off by it. The centre of the semicircle is the same as that of the circle.

19. Rectilineal figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four and multilateral those contained by more than four straight lines.

20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal and a scalene triangle that which has three sides unequal.

21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle and an acute-angled triangle that which has its three angles acute.

22. Of quadrilateral figures, a square is that which is both equilateral and right angled, an oblong that which is right angled

but not equilateral, a rhombus that which is equilateral but not right angled, and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right angled. And let quadrilaterals other than that be called trapezia.

23. Parallel straight lines are straight lines which being in the same plane and produced indefinitely in both directions, do not meet one another in either direction.

A critical appraisal of Euclid. There can be no doubt about the profound contribution that the thirteen books of Euclid's Elements have made to human thought in general and geometry in particular. For over two thousand years, the intellectuals in all fields, politicians, statesmen, army generals, theologeans have been studying it to develop facility in logical reasoning. But, of late, there has been an enormous increase in the mathematical knowledge and it has revealed flaws and weaknesses in Euclid's postulates, axioms and definitions, some of which are as follows :—

(1) Euclid's definitions have been found to be defective. Instead of beginning with some undefined terms, he attempted to define everything. He made use of words like 'part', 'evenly', 'inclination', 'length', 'breadth', extremities, etc. which need to be defined.

(2) It is essential that set of axioms and postulates taken in any axiomatic system should be adequate which is not the case with Euclid. A case in point is his failure to recognise the necessity of ordering the points on the line or making assumptions which may have made it possible to distinguish as to which one of any three points on a line lies between the other two. In other words, his postulates do not provide any means for defining the concept of 'betweenness'. This serious omission was discovered by C. F. Gauss in 1835. M. Pasch in 1880, suggested a correction to it by supplying a postulate which is known as Pasch's postulate. As this concept will not be applicable to a 'closed curve' like a circle, by postulating about the concept of 'betweenness', we eliminate the possibilities of a line having the properties of a closed curve.

(3) Euclid made use of a number of properties which he failed to include in his axioms. For example, he tacitly assumed that—

- (*a*) A line is of infinite extent.
- (*b*) The space is a continuous aggregate of points. In other words that space is not a discrete aggregate. This concept of continuity or denseness implies one-to-one correspondence between the points on a line and real numbers. He neither postulated about real number system nor did he set up a correspondence between the points in space and real numbers.
- (*c*) If there are two unequal linear segments there exists a finite multiple of the shorter segment which is longer than the other (Archimedean postulate).

(*d*) The arcs that are drawn to construct an equilateral triangle (First Proposition of Euclid) on a given segment as base do intersect. In case the points on a circular arc are spaced like beads on a thread then the arcs will not intersect but Euclid did not take up under what condition two circles will intersect although he took up the condition under which two straight lines will intersect. (The postulate mentioned against serial number 5 earlier.)

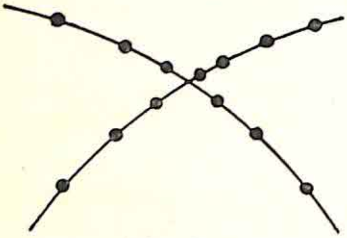

Fig. 4.

(4) Euclid introduced the concept of 'superposition' of the line segments and angles by alluding to the idea of 'motion of rigid bodies'. If a point has a location, how can a point be moved and still be the same point ? Two congruent triangles can be drawn on a surface of a sphere but if we attempt to draw two congruent triangles on an egg shaped body at two different parts of its surface with three pairs of equal sides then the pairs of angles of two triangles will not be equal. The sum of the angles of the triangle drawn at the sharper pole of the body would be less than two right angles, while it may not the be case when one is drawn at the blunter pole or the equator. Hence on such a surface a triangle cannot be moved without a change of form. Hence it clearly indicates that superposition is not only logically unsound and defective but also completely misleading. In fact, the foundation of all proofs given by Euclid consists in establishing the congruence of lines, angles, plane figures, solids etc. To make the congruence evident the geometrical figures are supposed to be applied to one another without changing their form and dimensions. But this may not be possible unless the surface is that of a sphere or that of a plane paper.

(5) Geometers have been constantly challenging the validity of the Euclid's postulate mentioned against serial number 5 and which is commonly known as Euclid's parallel postulate. Russian N.I. Lobatchevski and Hungarian John Bolai developed a new geometry by adopting a postulate which is quite contradictory to Euclid's parallel postulate and developed a new geometry known as hyperbolic geometry—a deductive system which is consistent within itself. Similarly, G.B.F. Riemann took yet another alternative of this postulate and developed another geometry—another consistent deductive system. Moreover, these geometries have been found to have applications in Physics. These two geometries have been taken up in some detail at a later stage. However, it may be pointed out here that inclusion of this postulate by Euclid does not lead to any defect in the deductive system that he developed.

(6) As at the time of Euclid, adequate algebraic symbolism and techniques had not been developed, his proofs lack that refineness which can be achieved by using algebraic symbolism and techniques.

(7) The proofs of theorems are sometimes made to depend on an appeal to the feeling of self-evidence or to the characteristics of geometrical figure. In axiomatic method, an appeal explicit or implicit to the feeling of self-evidence, to the characteristics of geometrical figures or to our experience concerning the behaviour of rigid bodies or the like is strictly prohibited. We may seek the help of such devices in getting a guidance to the steps we should take to build up the proof but the proof itself must not contain any reference to these aids. The conventional geometry course in giving a proof regarding the point of concurrency of medians of a triangle as point of trisection of each median would in general prove the following points :—

(1) ED is parallel to BC and is half as long as BC.

(2) △ EGD is similar to △ BGC.

(3) EG is half as long as GC and GD is half as long as BG.

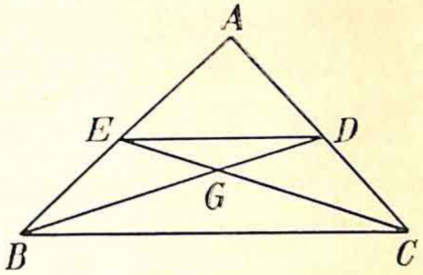

Fig. 5.

In this procedure, it has been taken for granted that the point of concurrency, G, lies between B and D and between C and E. This assumption is purely based on geometrical intuition and it cannot be deduced from Euclid's postulates. Hence the proof is made to depend on a reference to intuition and characteristics of geometrical figure which is not allowed in a postulational system.

These weaknesses in Euclid are mainly responsible for giving a 'logical' proof of such paradoxes as—

(1) Prove that every triangle is isosceles ; or

(2) Prove that every obtuse angle is a right angle ; or

(3) Prove that every point that lies in the interior of a circle lies on the circle ; etc. etc.

We know that these are incorrect statements as every triangle cannot be isosceles or an obtuse angle can never be a right angle or a point will be either in the interior of the circle or on the circle but it cannot be both. We give below the proofs of first two and challenge the reader to find the defect. Of course, the proofs are given in conventional form.

Theorem. Prove that every triangle is isosceles (a false statement).

Given. An arbitrary triangle ABC.

To Prove. AB=AC.

Construction. Draw AO, the bisector of ∠BAC and, DO, the perpendicular bisector of BC and let them meet at O. Draw OE perpendicular to AB and OF perpendicular to AC. Join OB and OC.

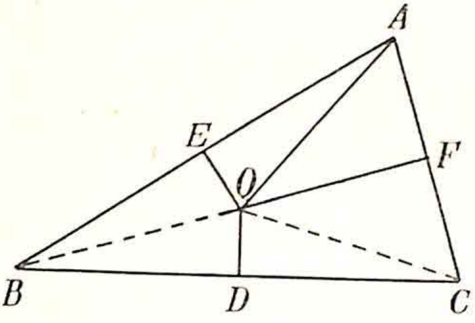

Fig. 6.

Proof.

(1) BD=DC. (Const. OD is the perpendicular bisector of BC)

(2) ∠ODB=∠ODC=90°.

(3) OD=OD.

(4) △ ODB is congruent to △ ODC. (S.A.S.)

(5) Hence OB=OC.

(6) △ AEO is congruent to △ AFO as AO is common, each of angles AEO, AFO is a right angle and ∠EAO=∠FAO. (As AO is the bisector of the ∠BAC.)

(7) Therefore, AE=AF
and OE=OF. (Proved)

(8) △ OEB is congruent to △OFC
as OE=OF,
OB=OC (Proved)
and as each of the angles OEB and OFC is a right angle.

(9) Therefore EB=FC.

(10) AE=AF (Proved)
Hence AE + EB=AF+FC
or AB=AC.

Hence the triangle ABC is isosceles.

Now the reader is challenged to find the defect. He may have to decide that if △ ABC is not isosceles then what will be the position of O and how does he know about it.

Theorem. Prove that every obtuse angle is a right angle.

Given. An arbitrary obtuse angle ABC.

To Prove. $\angle$ABC is a right angle.

Construction. Let D and E be points on the same side of BC as A and let BD=BA.

Let BCED be a rectangle. Let the perpendicular bisectors of AE, and BC meet in O.

Proof.

(1) OE=OA. ($\because$ O lies on perpendicular bisector of AE)

(2) OB=OC. ($\because$ O lies on perpendicular bisector of BC)

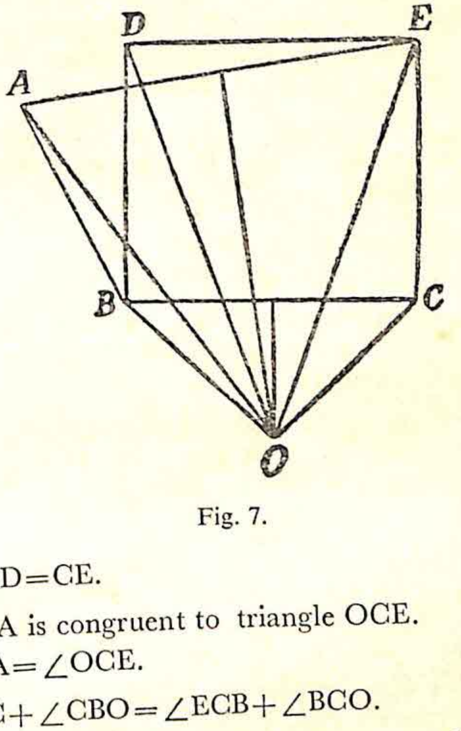

Fig. 7.

(3) AB=BD=CE. (Construction)

(4) $\triangle$ OBA is congruent to triangle OCE.

(5) $\angle$OBA=$\angle$OCE.

(6) $\angle$ABC+$\angle$CBO=$\angle$ECB+$\angle$BCO.

Now $\angle$CBO=$\angle$BCO as O lies on perpendicular bisector of BC. Hence $\angle$ABC=$\angle$ECB. But $\angle$ECB is a right angle by construction. Hence $\angle$ABC is a right angle.

Here is yet another challenge for the reader to find out defect in its proof.

It was David Hilbert, a German mathematician, who was the first to remove these defects in his book 'Foundations of Geometry' which he published in 1899. He built up a postulational system and the approach that he employed has since become a classic. His system consisted of the following :—

(*a*) *Five undefined terms*—(1) Point, (2) Line, (3) On, (4) Between, (5) Congruent.

(*b*) Fifteen postulates including six postulates on congruence to overcome the difficulties of 'rigid motion' and 'superposition' ; parallel postulate, four postulates dealing with the concept of order and betweenness, two postulates dealing with the concept of continuity and two postulates of connection. As we shall be employing Hilbert's approach regarding betweenness and continuity as we develop the system in this book, it will not be out of place to list them down.

Hilbert's Postulates of Betweenness and Continuity

Betweenness. (1) If point C is between points A and B, then A, B, C are on the same line and C is between B and A and B is not between C and A and A is not between C and B.

(2) For any two distinct points A and B, there is always a point C which is between A and B and a point D which is such that B is between A and D.

(3) If A, B, C are three distinct points on the same line, then one of the points is between the other two.

(4) A line which intersects one side of a triangle but does not pass through any of the vertices of the triangle, must also intersect another side of the triangle.

Postulates that Hilbert adopted for the 'concept of continuity' could be replaced by a single postulate proposed by Richard Dedekind, also a German mathematician, which can be stated in the following words :—

"If all points of a straight line fall into two sets such that every point of the first set lies to the left of every point of the second set, there exists one and only one point which creates this division into two sets."

Hilbert's Postulates of Connection.

(1) Two non-coincident points determine a straight line.

(2) There are atleast four points that do not lie in one plane.

We propose to develop in this book, synthetic geometry as an abstract deductive system and for this we shall make use of the following :—

(1) We will make use of language of set theory and laws of elementary logic for deducing new statements.

(2) We will make use of some undefined terms like point, line, plane, number and relations like on, between, similar, equal, and congruent. We will make use of internationally recognised symbols for these undefined terms and relations.

(3) A set of 24 postulates characterising the undefined terms. We shall formulate our postulate in such a manner that the Euclid's weaknesses pertaining to 'betweenness', and superposition of figures are removed. We will also make provision for the 'concept of continuity' by setting up one-to-one correspondence between the points on a line and the set of real numbers. We shall take care to see that our set of postulates is self-consistent.

(4) We will make use of a number of definitions and we will define them as and when we need them.

(5) We will build up a body of theorems with the help of undefined terms and relations, definitions, and assumptions. While doing so we will take care to see that—

(*a*) We do not tacitly assume properties on the basis of figure or spatial intuition.

We shall make use of figures just to help us to see the relationships. We will not make our proofs depend on the characteristics of figures.

(*b*) We will not make use of any relation howsoever familiar it may be unless we have deduced it from previously established or assumed relation.

All these precautions are essential for removing the weaknesses of Euclid.

It will however not be out of place to mention the definition of Mathematics as given by Bertrand Russell. "Mathematics", Bertrand Russell says, "is that subject in which we never know what we are talking about nor whether what we are saying is true." Apparently, this definition gives the impression that Mathematics is a useless subject as according to Russell, neither we know what we are talking about nor we know about the truth of what we talk about in Mathematics. But if we try to read between the lines, we will find that Russell's definition truly portrays the nature of Mathematics. In fact, in Mathematics, the mathematician does not know what he is talking about. When a mathematician says that two plus three make five, he does not know whether he is talking about two and three apples or something else. The statement 'we never know what we are talking about, implies the abstract nature of Mathematics. The development of Geometry in Egypt had reference to plots of land and the physical devices that can be used to delineate those but in the hands of Greeks, it underwent a big transformation and rather than talking about plots of land, the Greeks started talking about abstract surfaces and volumes bounded by abstract lines. To the

Greeks, a triangle could be a 'triangle of land', or 'triangle of sea', or 'triangle of space', or 'triangle of pastry', etc. etc. It proved to be more useful to talk about 'triangles' rather than 'triangles of land' as the operations on abstract triangles could be applied equally well to triangles of 'land' or 'sea' or 'pastry'. Greeks were the first to recognise the value of saying something when one does not know what one is talking about and this continues to be the characteristic of Mathematics today.

We also do not know whether what we are saying in Mathematics is true or not. Earlier we made a mention of an example wherein a person had booked the air passage to some place for a particular day but he had to think of many other alternatives as there came up the possibility of the flight being cancelled. So he had to think about all other possibilities and their consequences although he never knew what course of action he would have to adopt on the day he was to travel. So in Mathematics we start with some initial assumptions and derive results without bothering very much to know as to which set of assumptions are going to hold good in a particular situation. While building up synthetic geometry as an abstract deductive system we will keep in view the guidelines given by Bertrand Russell. Before doing so, we would like to talk in some detail about inductive and deductive reasoning, set theory and its language, laws of logic, proof, real numbers, etc. etc., as we are going to make use of these in developing our system.

Inductive Reasoning. As already stated, the geometry that was developed by the Egyptians was based on inductive reasoning. Suppose you draw quite a number of triangles and then find the sum of angles of each of the triangles by actual measurement and you find that in some cases it is little less than 180° while in some other cases it is little more than 180°. You conclude from this experiment that on an average, the sum of angles of a triangle is 180°. Then you state a rule that the sum of angles of a triangle is 180°. Now this rule may be true or may not be true as you have derived the rule from consideration of many special cases and you have not supported it by a logical proof. Reasoning of this type is known as inductive reasoning.

Supposing you draw a pair of intersecting lines, measure the vertical angles and find that vertical angles are equal. You repeat this experiment a number of times and in every case you find that the vertical angles are equal. You state the rule that 'a pair of vertical angles are equal'. But it would be better to insert the word 'probably' and to state that 'probably a pair of vertical angles are equal' as till you have supported your observation by a logical proof you are not sure whether, if you repeat the experiment, you will not come accross a pair of vertical angles which may be unequal.

From the discussion given above, you might have got the impression that inductive reasoning is primitive and not very reliable and it is not very useful for Maths. and Sciences. But this is not so. Inductive reasoning has made a very large contribution to the development of our present day civilization. It has been extensively used to develop principles and laws in Physics, Chemistry, Biology, Astronomy, Medicine, Agriculture, Psychology, Sociology, Economics etc. Although in the present course we will not have to do very much with inductive reasoning yet it is very important to know about it as Geometry had its beginning as a body of empirical knowledge.

As already pointed out, inductive reasoning is a process in which quite a number of particular cases are observed and on the basis of these, generalizations in the shape of laws, relations or properties are made. Given below are some exercises which need the use of inductive reasoning on your part. Try to be very careful in making your measurements and generalizations. Be intellectually very honest and do not cook up the results just because the measurements and generalizations that you make do not agree with some of the properties or laws that you have already studied. You have to simply forget that you know any properties of geometrical figures.

EXERCISES

1. Draw several triangles and find the sum of angles of each of these. What appears to be the sum of angles of each ? Are you absolutely sure that you have arrived at correct generalization ?

2. Repeat the above experiment by drawing several quadrilaterals. What generalizations can you make from the results that you arrive at ?

3. Draw several triangles and measure their sides. Find the sum of lengths of two shorter sides in each case. Is this sum greater than the length of the largest side in the case of each triangle ?

4. Draw several circles and in each circle draw one of its diameters. In each case join the end points of the diameter to a point on the circle and measure the angle so formed. What generalization can you make regarding this particular angle ?

5. Draw several circles and in each circle draw inscribed quadrilaterals by taking four points on each circle. Measure the angles of each inscribed quadrilateral and find the sum of opposite angles in each case. Can you draw any generalizations ?

6. Draw several triangles and measure the sides and angles in each case. On the basis of this experiment can you say that in each case, the longest of three sides has opposite to it the largest of the three angles and the smallest of three sides has opposite to it the smallest of the three angles ?

7. A farmer has sown wheat in two adjoining fields which are similar to each other. In one field, he makes use of a chemical while

in the other he does not make use of any chemical. He observed that the field in which chemical was used gave better results as compared to the other field. Next year he repeats the experiment and gets the same results. Can he conclude that :

(*a*) The use of chemical will increase the yield of wheat in every case ?

Or

(*b*) There is high degree of probability that the use of that chemical will increase the yield of wheat ?

Or

(*c*) There is a probability that the use of that chemical will increase the yield of wheat ?

8. Construct a quadrilateral of any shape and find the mid-points of its sides. Repeat this experiment a number of times. What can you state about the figure formed by the join of the mid-points ?

Weaknesses in Inductive Method

In inductive method we observe some particular instances and then infer general properties, relations or laws. Inductive reasoning suffers from two very serious weaknesses which are as follows :

(1) Our measurements and observations can never always be absolutely accurate.

(2) We can never exhaust all the possible cases before making generalizations. There is ever a possibility of some exceptional case which may prove the generalization false. There is a classical example. Mathematicians observed that the absolute value of coefficients of factors of the binomial x^n-1 never exceeded one. They had generalized it as they found it true in the case of the binomials (x^2-1), (x^3-1), (x^4-1) etc. etc. This view was held for centuries but later on it was proved that it was true upto $x^{104}-1$ but failed in the case $x^{105}-1$ as $x^{105}-1$ has a factor

$$x^{48}+x^{47}+x^{46}-x^{43}-x^{42}-2x^{41}-x^{40}-x^{39}+x^{36}+x^{35}+x^{34}$$
$$+x^{33}+x^{32}+x^{31}-x^{28}-x^{26}-x^{24}-x^{22}-x^{20}+x^{17}+x^{16}+x^{15}$$
$$+x^{14}+x^{13}+x^{12}-x^{9}-x^{8}-2x^{7}-x^{6}-x^{5}+x^{2}+x+1$$

as in this case the absolute value of coefficient of each of x^{41} and x^7 is 2. This proves that unless we exhaust all the possible cases, there is ever a possibility of some exceptional case which may prove the generalization false.

Let us consider another example. Let us find the value of the trinomial x^2-x+17 for all x. We take x in turn equal to 1, 2, 3...10 and tabulate the value of x^2-x+17 as follows :—

x	x^2-x+17	x	x^2-x+17
1	17	9	89
2	19	10	107
3	23	11	127
4	29	12	149
5	37	13	173
6	47	14	199
7	59	15	227
8	73	16	257

On observing the values of x^2-x+17, we find that these values have the following properties :

(1) The value is an odd number.

(2) The value is a number which ends either in 3 or 7 or 9.

(3) The value is a prime number.

So we are prone to make a rule the the value of the expression x^2-x+7 for all x is a number which is always odd, which ends either in 3 or 7 or 9 and which is a prime number. But are we justified in framing this rule on the basis of only a few values of x ? What about the value of x^2-x+17 when $x=17$? It is 289 which is not prime as $289=17\times17$. So our rule fails in this respect that its value is always a prime number. Another example is that of the expression x^2-x+41. This expression gives a prime number if x is taken as 1, 2, 3...40. When we take $x=41$, the number that we get is not prime. In the present course, we will not be very much concerned with inductive reasoning. Instead, we will be making use of deductive reasoning which is opposite to inductive reasoning. Induction helps us in framing the hypothesis and this is put to test by deducing from them the predictions to be confronted with the testimony of future experience.

Deductive Reasoning

Deductive reasoning can be described as an operation of mind 'inverse to' that of inductive reasoning. Induction leads us from less general to more general propositions while deduction helps us in leading to conclusions of generality either equal to or less than its premises. In inductive reasoning, we observe some specific case and on observing some common property, we make a general statement. In deductive reasoning we start by making general statements and then apply it to specific cases. Any case of deductive reasoning will mainly consist of following three steps :—

(*a*) A general statement.

(*b*) A specific statement.

(*c*) The conclusion.

Some examples of deductive reasoning are the following.

Example 1.

General statement. All animals have four legs.

Specific statement. Cow is an animal.

Conclusion. The cow has four legs.

Now if we analyse the problem, we find that the conclusion that we have drawn is logically correct. You may point out that the cow is an animal which has four legs. Why to talk about a general statement, a specific statement and a conclusion? To clarify the matters let us consider the following example.

Example 2.

General statement. All animals have three legs.

Specific statement. Cow is an animal.

Conclusion ?

Now what will be the conclusion that you draw from this proposition ? The cow has four legs ? Certainly not. The logical conclusion would be : The cow has three legs. So it may now be clear that the conclusion drawn has to depend on 'general statement' and in actual life situation, the conclusion drawn is true with reference to physical world only if the general statement is true with reference to the physical world. Deduction is absolutely not concerned with the truth or the otherwise of the general statement with reference to physical world. This is inherent weakness in deductive reasoning. Deductive reasoning only asserts that 'something is true if something else is true' but it does not concern itself with the truth of something else.

Now the big question that we need to consider is : "What is truth ?' If by truth or the quality of being true we mean that it should have measure of correspondence with the real world then the plain fact is that there is nothing like 'truth' to be found in Mathematics. If you ask whether $2+2=4$ is true, the reply that a mathematician would give would be that $2+2=4$ follows from the assumptions and definitions he has made regarding the numbers two and four and the binary operation of addition. So with reference to these assumptions and definitions the statement $2+2=4$ is true and if there is something like 'absolute truth' then he does not know about the 'absolute truth' of $2+2=4$. If you ask whether two apples plus two apples make four apples then the reply that a mathematician would give would be that he does not know as this is a question that does not relate to Mathematics. Of course, he would add that as a common man he would say that two apples together with two more apples do make four apples.

This 'inherent weakness' in deductive reasoning gives a very big advantage to the mathematician as there may not be anything like 'absolute truth'. In case you make a statement that you drove your car at 60 miles per hour toward east then will this statement be 'absolutely true'? It may be true relative to the earth being considered stationery. But far from being at rest, the earth is whirling round the sun at the rate of 20 miles a second. Simultaneously, it rotates about its axis at the rate of 1000 miles an hour. The earth is a part of the solar system and the entire solar system is moving within the local star system at the rate of 13 miles a second. The local star system is moving within the milky way at the rate of 200 miles a second. The whole milky way is drifting with respect to external galaxies at the rate of 100 miles a second. Moreover, all these motions are in different directions. Now can you say that your statement is absolutely true ? You can only say that it is true if something else is true.

We can supplement our argument by taking up an example. In Euclidean geometry (including the famous parallel postulate), if we make the statement that the sum of angles of a triangle is two right angles, then it will be a true statement with reference to the Euclidean system. In fact, we should use the word 'consistent' rather than using the word 'true' and state that the statement made by us is consistent within the system built by Euclid. Now if we remove the Euclid's parallel postulate and substitute it with an alternative parallel postulate then we can make the statement that the sum of angles of a triangle is always less than two right angles and this statement will be true or consistent within the system under consideration. Now by substituting yet another parallel postulate, we can make the statement that the sum of angles of a triangle is always greater than two right angles and this statement will be true within this new system. The mathematician only knows whether a statement is true (or consistent) or not within a particular system and he does not bother to know whether it is true with reference to the physical universe or not. He leaves it to a physical scientist to find out. But alas, even the physical scientist does not know which one of the statements that we have made regarding the sum of the angles of a triangle is true with reference to the physical universe we live in.

Now let us study an example in which the conclusion through deductive reasoning is faulty.

Example 3.

General statement. All animals have four legs. Cow is an animal.

Specific statement. x is an animal which has four legs.

Conclusion. x is a cow.

It is obvious that the conclusion derived is not logically correct.

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It may however be pointed out that the three stages involved in the deductive reasoning may be used in any order which we may like.

An example is given here :

Specific statement. Rani is a student of Government Girls School.

Conclusion. Rani has to study home science.

General statement. All students of Government Girls School have to study home science.

The reasoning is valid as the conclusion has been logically deduced from the general statement.

We can define deduction as a rigorous proof or derivations of the statement (the conclusion) from one or more statements [the premises or the general statements or the postulates or the statements that are given to us which we accept without any proof—which go by the term 'Given' in a conventional course of geometry—a chain of statements each of which is either a premise (or given) or follow a statement occurring earlier in proof]. Through deductive reasoning we derive conclusions which are either equal to or less than that of its premises. In other words through deductive reasoning we derive conclusions which are contained in the postulates that we have accepted. We cannot conclude anything more than what is built in, in the postulates, definitions, etc.

In geometry we deal with the statements of 'If—Then' form. It is the inductive reasoning that suggests us the 'If-part' of the statement and through deductive reasoning we prove the 'Then-part' of the statement. Without induction, deduction has nothing to prove and without deduction it is not possible to test the validity of the generalizations arrived at through induction. It is the induction that suggests the theorems and it is the deduction that helps us in proving them. Without deduction, induction is unsure of its inferences and liable to be disproved by a counter-example. A Mathematician, therefore, has to have skill in both induction and deduction or in other words, he needs both the intuition as well as a clear feeling for the proof.

So far we have been discussing inductive and deductive reasoning which are important constituents of 'Elementary Logic' which we would need to build up our system. Before we take up something more about 'Logic', it will be very useful to acquaint ourselves with very very elementary 'Set Theory' and the symbols that it employs.

Given below are some problems on deductive reasoning for you to try your hand at.

Problem Set

I. State the conclusion that you arrive at with the help of general statement and the specific statement in each case.

(*a*) *General statement.* In Government Girls School, only girls are admitted.

Specific statement. A is a student of Government Girls School.

Conclusion ?

(*b*) *General statement.* To be a member of the Union Basketball Team, the player has to be more than 170 cm. tall.

Specific statement. Sudhir is a member of the Union Basketball Team.

Conclusion ?

We can give another form to this very problem which is as follows :

To be a member of the Union Basketball Team, the player has to be more than 170 cm. tall. If Sudhir is a member of Union Basketball Team, prove that Sudhir is more than 170 cm. tall.

(*c*) *General statement.* The sum of the angles of a triangle is 180°.

Specific statement. ABC is a triangle.

Conclusion ?

(*d*) *General statement.* The sum of the angles of a quadrilateral is 360°.

Specific statement. ABCD is a quadrilateral.

Conclusion ?

II. State whether the conclusions drawn in the following examples are valid. Support your answers with arguments.

(*a*) *General statement.* In Lady Irwin School, only the students of Kamla Nagar locality are admitted.

Specific statement. Kamla is a girl student.

Conclusion. Kamla is a student of Lady Irwin School.

(*b*) *General statement.* Ramu rides back home from the school on the School Bus.

Specific statement. Ramu is riding on a bus.

Conclusion. Ramu is coming back home from school.

(*c*) *General statement.* If equals are divided by equals the quotients remain equal.

Specific statement. $3x=51$

Conclusion. $x=17$.

(*d*) *General statement.* If equal quantities are added to both sides of an equation, the equation remains unchanged.

Specific statement. $x-7=13$
Conclusion. $x=20$

(*e*) All girls who weigh more than 200 kilograms are pretty.
Specific statement. Sussie weighs more than 200 kilograms.
Conclusion. Therefore Sussie is pretty.

(*f*) *General statement.* All mathematicians work very hard.
Specific statement. Neena is a mathematician.
Conclusion. Neena works very hard.

(*g*) *General statement.* All mathematicians are very helpful.
Specific statement. Kathy is very helpful.
Conclusion. Kathy is a mathematician.

III. Wherever possible draw valid conclusions from each of the following pairs of general and specific statements. If a valid conclusion is not possible, explain why it is not possible.

(*a*) *Specific statement.* ABC is a triangle and the line segment DE joins the mid-points of the sides AC and BC.

General statement. If a line segment joins the mid-point of two sides of a triangle, it is one half of the third side.

(*b*) *General statement.* When Ram goes to school, he rides on a school bus.

Specific statement. Ram is riding on a bus.

Sets and Their Language

The language and notation of sets is very handy in making definitions precisely and writing the proofs concisely. We need to understand the meaning of the terms like sets, subsets, union of sets, intersection of sets and the notation that is used in expressing them. It is presumed that you have a little knowledge about sets.

A *set* is a collection of objects. Generally, it is a well defined collection of discrete objects like the students in ninth class, desks in a class room, books on a shelf, the points on the line PQ, all men in a village, all girl students in the university or all teachers in a school. The objects in the set may consist of physical things, symbols, facts or ideas. On occasions we use the terms like herd, class, assemblage, aggregate, manifold, cluster, flock etc. We can replace these terms by a common term 'set' as all these terms convey the idea of a collection of objects or things. If a pen, a pencil and a fork are lying on a table, it will be a set. Some examples of sets are the following :

$S_1=\{\text{pen, inkpot, pencil}\}$
$S_2=\{\text{circle, triangle, line, point}\}$
$S_3=\{\triangle, \#, \neq, \square, \odot\}$
$S_4=\{a, b\}$
$S_5=\{x, y, 1, 2\}$
$S_6=\{1, 2, 3, 10, 7\}$ etc. etc.

The individual objects that make a set are called the elements of the set or members of the set. In the sets that we have mentioned, pen is an element of S_1 and we denote it as : pen $\in S_1$. $\triangle$ is an element of S_3 and we denote it by writing $\triangle \in S_3$. Also in these sets, $x \in S_5$; $10 \in S_6$ and so on. We also say x belongs to S_5 or 10 belongs to S_6. Element of, belongs to, or member of, are equivalent phrases for this purpose. Notice that elements of a set are written within braces and each element is separated by a comma.

How to designate a set. There are two ways of designating a set.

(1) List all the names of the members of the set.

Or

(2) To describe them by some common property.

Supposing we have to write a set S_1 containing the elements 2, 4, 6, 8, 10, 12, 14. We can write it as S_1 : {2, 4, 6, 8, 10, 12, 14} or we can designate it by writing 'all even natural numbers less than sixteen' or 'all even natural numbers upto fourteen'. The second method proves very helpful generally as you can describe a set containing large number of elements very concisely. Examples are : Set of whole numbers divisible by 3 or the set of all triangles or the set of all the lines in a plane through a point or the set of all the angles inscribed in the arc of a circle or the set of all the points in the interior of a triangle etc.

Empty set. It is clear that a set can have any number of elements. Sometimes a set may have no elements at all. For example, the set of all the girls with three eyes or the set of real roots of the equation $x^2+1=0$ or the set of the points common between two coplanar parallel lines. A set that does not contain any element is called an empty set or a null set and it is denoted by the symbol ϕ which is a letter of Greek alphabet and which is read as 'phai'. { } also indicates an empty set.

A set may have only one element such as 'the set of mid-points of a line segment', or two elements such as 'the set of diagonals of a square' or three elements such as 'the set of vertices of a triangle' etc.

A set can be infinite. The set of the points in the interior of a triangle would be such a set.

In the foregoing we have not tried to define the terms set or element. In fact we would like to take them as our undefined terms. Similarly, we will take the statements 'belong to' or 'element of' or 'member of' as an undefined relation.

The Universal Set and Subsets

It is also essential to know about the terms universal set, subset etc. An infinite set such as the set of all points in a plane is generally called a 'universal set'. Then we can have special sets within the universal set. Supposing our universal set is 'the set of all triangles', then the set of all equilateral triangles or the set of all isosceles triangles will each be a special set within this universal set. We often make use of Venn diagrams to make its visual representation. It may be clearly understood that Venn diagram is only an aid. Figure 9 is the representation of the example taken above, by a Venn diagram.

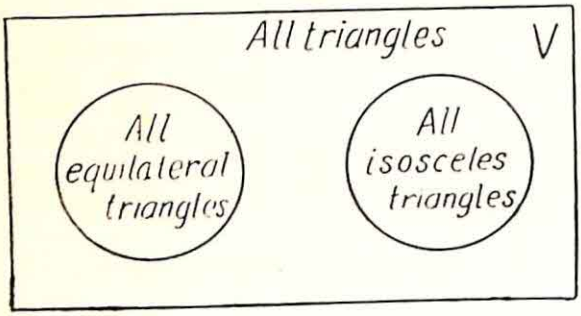


Fig. 9. Venn diagram.

Now it is clear that every element of the special set 'all equilateral triangles' is also an element of the universal set of 'all triangles'. In such a case it is said that 'the set of equilateral triangles' is a subset of the 'set of all triangles'. If U is our universal set and E is a subset of U, then every element of E will be an element of U. Symbolically, we write it as, for all *t's* if $t \in E$ then $t \in U$. With reference to the example we have taken, we have only said that for all triangles, if a triangle is an element of the set of equilateral triangles then that triangle is an element of set of all triangles and the 'set of equilateral triangles' is a subset of 'set of all triangles'. If E is a subset of U then we write it as $E \subset U$ or $U \supset E$. We can also read them as : the set E is contained in the set U or the set U contains the set E.

If A, B are any two sets such that for all x, if $x \in A$, then $x \in B$ and then $A \subset B$ or $B \supset A$. Supposing we have a set $S : \{x, y, z\}$, then following will be the subsets of S:—

(1) $S_1 : \{x, y, z\}$
(2) $S_2 : \{x, y\}$
(3) $S_3 : \{y, z\}$
(4) $S_4 : \{x, z\}$
(5) $S_5 : \{x\}$
(6) $S_6 : \{y\}$
(7) $S_7 : \{z\}$
(8) $S_8 : \{\ \}$ or ϕ

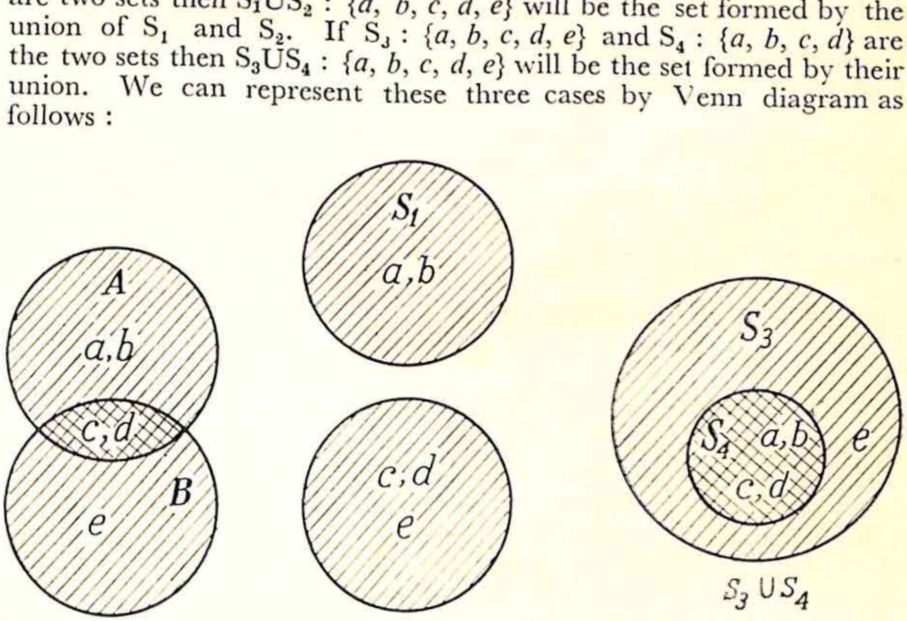
You may have an objection to our including $S_1 : \{a, b, c\}$ and $S_8 : \{\ \}$ in the list of subsets of the set S. But S_1 satisfies the definition of 'subset' as every element of S_1 is an element of S. In other words S_1 does not contain any element which is not an element of S. Also the mathematicians have decided that ϕ would be a subset of every set.

Construction of New Sets

We often make use of sets that we construct by taking two or more sets. Supposing we have two sets A and B, then we can construct three different types of sets with the help of these two : (1) By forming their union ; (2) By forming their intersection ; and (3) By forming their product.

Union of Two Sets

If A and B are any two sets then their union will be a new set whose members are either the members of A or B or both. The new set is denoted by AUB which is read as 'A union B'. If A : $\{a, b, c, d\}$ and B : $\{c, d, e\}$ are the two sets then AUB : $\{a, b, c, d, e\}$ will be the new set. Now every member of AUB is either the member of A or B or both. It may be understood that there is no need of repeating the elements in any set. It will be incorrect to write AUB : $\{a, b, c, d, c, d, e\}$. If $S_1 : \{a, b\}$ and $S_2 : \{c, d, e\}$ are two sets then $S_1US_2 : \{a, b, c, d, e\}$ will be the set formed by the union of S_1 and S_2. If $S_3 : \{a, b, c, d, e\}$ and $S_4 : \{a, b, c, d\}$ are the two sets then $S_3US_4 : \{a, b, c, d, e\}$ will be the set formed by their union. We can represent these three cases by Venn diagram as follows :

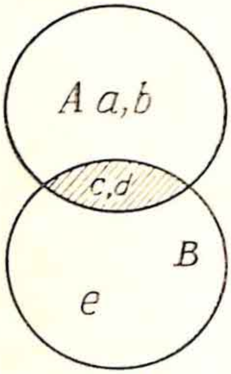


AUB $\qquad S_1US_2 \qquad S_3 \cup S_4$

Fig. 10.

Intersection of Two Sets

Also we can form a new set by forming an intersection of two sets. If we form a new set containing all those elements that belong to both the sets then this will be the intersection of two sets. In other words, the intersection of two sets contains all the elements common to two sets.



$A \cap B$
(Shaded part)
Fig. 11.

If A, B are any two sets then their intersection is written as $A \cap B$. Following are the examples of the new sets formed by the intersection of two sets :

(1) A : $\{a, b, c, d\}$;

B : $\{c, d, e\}$

then $A \cap B : \{c, d\}$

(2) $S_1 : \{a, b\}$;

$S_2 : \{c, d\}$

then $S_1 \cap S_2 : \{\ \}$

as there are no elements common between S_1 and S_2.

(3) $S_3 : \{a, b, c, d, e\}$;

$S_4 : \{a, b, c, d\}$

then $S_3 \cap S_4 : \{a, b, c, d\}$.

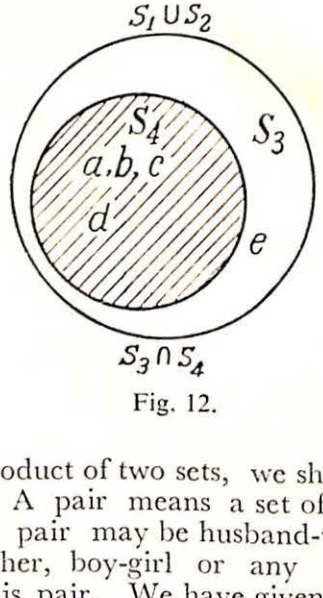


Fig. 12.

Cartesian Product of Two Sets

To understand the idea of the product of two sets, we should first know what is an ordered pair. A pair means a set of two things. {Hari, Meena} is a pair. This pair may be husband-wife, brother-sister, father-daughter, son-mother, boy-girl or any such thing. We have given an 'order' to this pair. We have given first place to Hari and second to Meena. So this is an ordered pair. Now (a, b) is an ordered pair. Also (b, a) is an ordered pair. Now if a is not equal to b or $a \neq b$ then (a, b) is not the same as (b, a) or $(a, b) \neq (b, a)$. If $a=b$ then of course $(a, b)=(b, a)$. If (a, b) and (c, d) are any two ordered pairs then $(a, b)=(c, d)$ if and only if (or iff) $a=c$ and $b=d$. In an ordered pair, the member that is written to the left is the first member and that which is written to the right is the second. If $S_1 : \{a, b\}$ and $S_2 : \{x, y, z\}$ then we can form following ordered pairs by taking the first element always from S_1 and the second element always from S_2 :

(a, x) ; (a, y) ; (a, z) ; (b, x) ; (b, y) ; (b, z)

Then the product or cartesian product of S_1 and S_2 written as $S_1 \times S_2$ will be the set formed by taking all these ordered pairs. In other words,

$$S_1 \times S_2 = \{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)\}$$

Also we need to know what are equivalent sets and identical sets. If there are two sets A and B :—

$$A = \{a, b, c\}$$

and $$B = \{x, y, z\}$$

Then these two sets are such that to each member of A, there corresponds one and only one member of B and to each member of B there corresponds one and only one member of A, or in mathematical terms if two sets can be uniquely paired or if one-to-one correspondence could be set up between the elements of two sets, then these sets are said to be equivalent. If A and B are two equivalent sets then we write that $A \leftrightarrow B$. We can represent this idea as follows in six different ways :

$$\begin{tikzcd} A=\{a, \arrow[d] & b, \arrow[d] & c\} \arrow[d] \\ B=\{x, & y, & z\} \end{tikzcd}$$

or

$$\begin{tikzcd} A=\{a, \arrow[dr] & b, \arrow[dl] & c\} \arrow[d] \\ B=\{x, & y, & z\} \end{tikzcd}$$

or

$$\begin{tikzcd} A=\{a, \arrow[d] & b, \arrow[dr] & c\} \arrow[dl] \\ B=\{x, & y, & z\} \end{tikzcd}$$

or

$$\begin{tikzcd} A=\{a, \arrow[drr] & b, \arrow[d] & c\} \arrow[dll] \\ B=\{x, & y, & z\} \end{tikzcd}$$

Two equivalent sets are said to be identical if each element of one set is an element of the other set. A : $\{a, b, c\}$ and B : $\{b, a, c\}$ are identical sets and we write $A = B$. It is clear that if $A = B$, then $A \subset B$ and $B \subset A$.

Some Symbols and Idea of a Variable and its Domain

Some other symbols we should know are :

$\notin$, $\ni$, $\not\ni$, $\not\subset$, $\not\supset$, $|$, etc.

Now e is not an element of $\{a, b, c\}$. We write it as $e \notin \{a, b, c\}$. Also $\triangle ABC \notin$ of set of squares. Also we know that the set of all rectangles is not a subset of set of squares. We write it as set of all rectangles $\not\subset$ set of squares. We can also write it as set of squares $\not\supset$ set of all rectangles which means that the set of squares does not contain the set of rectangles.

The symbol $\ni$ stands for 'there exists' and the symbol $\not\ni$ stands for 'there does not exist'. It is also important to know the idea of a variable. If we consider a set of all natural numbers *i.e.* N : $\{1, 2, 3...\}$ we can say that N is a set of all x such that x is a natural number. Here we have used the letter 'x' as a variable. Supposing we have a set W consisting of all x such that x is a whole number and if we want to list all the members of the set then we will have to replace the variable x by each of the members of set of whole numbers *i.e.* $\{0, 1, 2...\}$. In such a case $\{0, 1, 2...\}$ will be the replacement set for the variable x or the domain of the variable or the values of the variable. We can describe a set with the help of a variable concisely. Supposing we have set E of all even natural numbers. We can designate it as $E : \{x \mid x \in \text{even natural numbers}\}$. We read it as E is a set of all x such that x is an element of even natural numbers. $\mid$ is the symbol for the phrase 'the set such that'.

Now we can make the definition of union and intersection of two sets as follows :

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ which we read as A union B is a new set formed of all x such that x is element of set A or x is element of set B.

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

We have also the symbols for the connectives 'and' and 'or'.

$\vee$ stands for the connective 'or' and $\wedge$ stands for the connective 'and'. By making use of these symbols we can write the union and the intersection of two sets as :

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

and

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

We will be very often making use of the concepts and terminology of the 'Set Theory' and it would be very essential to get a facility with their use.

Problem Set

I. If $S = \{3, 5, 9, 17, 0, 2, 14\}$ is the given set then which of the following statements is true ?

(*a*) $2 \in S$ (*b*) $0 \in S$ (*c*) $15 \in S$ (*d*) $2 \notin S$
(*e*) $17 \notin S$ (*f*) $10 \in S$ (*g*) $35 \in S$ (*h*) $02 \in S$

II. For each of the following sets determine if—

(*i*) $A \subset B$ or (*ii*) $A \supset B$.

(*a*) $A = \{3, 1, 9, 2, 5\}$; $B = \{1, 2, 3...\}$

(*b*) $A = \{0, 3, 5\}$; $B = \{3, 0, 5\}$

(*c*) $A = \{5, 6, 7...\}$; $B = \{1, 2, 3...\}$

(*d*) $A = \{0, 1, 2...50\}$; $B = \{1, 2, 3...50\}$

(*e*) $A = \{x \mid x \text{ is a teacher}\}$; $B = \{x \mid x \text{ is a teacher who earned more than Rs. 500 a month}\}$

(*f*) $A = \{2, 4, 6...\}$; $B = \{4, 8, 12...\}$

III. If $A \subset B$ and $B \subset C$, then will $A \subset C$ or $A \supset C$?

IV. Which of the following statements will be true for any sets A, B, C ?

(*a*) $(A \cup B) \cup C = A \cup (B \cup C)$.
(*b*) $(A \cap B) \cap C = A \cap (B \cap C)$.
(*c*) $A \subset A \cup B$. (*d*) $A \supset A \cup B$ when $B \subset A$.
(*e*) $A \subset A \cap B$. (*f*) $A \supset A \cap B$. (*g*) $A \subset A \cup B \cup C$.
(*h*) If $A \subset B$ then $x \in A \rightarrow x \in B$.
(*i*) If $A \supset B$ then $x \in A \rightarrow x \in B$.
(*j*) If $A \subset B$ then $A = A \cap B$.
(*k*) If $A = A \cap B$ then $A \subset B$.
(*l*) If $A \subset B$ then $A = A \cap B$.
(*m*) If $A = A \cup B$ then $A \subset B$.

V. If S_1 is the set of all boys in 9th class, S_2 is the set of all girls in 9th class and if S_3 is the set of all students in the school then what is :

(*i*) $S_1 \cup S_2$ (*ii*) $S_1 \cap S_2$ (*iii*) $S_3 \cup S_1$ (*iv*) $(S_3 \cup S_1) \cup S_2$
(*v*) $S_3 \cap S_1$ (*vi*) $S_3 \cap S_2$ (*vii*) $S_1 \cap (S_1 \cup S_2)$.

Describe these in your own words.

VI. Give two examples of :

(*a*) Well defined sets (*b*) sets which are not well defined (*c*) finite sets (*d*) infinite sets.

VII. Show that one-to-one correspondence can be set up between the set of letters of English alphabet and the set $\{1, 2, 3, 4, \ldots 26\}$. Show that this matching can be set up in more than one way.

VIII. Show that there are six possible ways of matching the elements of $S_1 = \{a, b, c\}$ and $S_2\{1, 2, 3\}$.

IX. Show that if $A \subset B$ and $B \subset A$, then $A = B$.

X. Show that one-to-one correspondence can be set up between all the natural numbers and the set of positive even integers.

XI. Show with the help of an example that

$$\text{if } A \leftrightarrow B \quad \text{and} \quad B \leftrightarrow C \quad \text{then} \quad A \leftrightarrow C.$$

XII. Show that $A \cup \phi = A$ and $A \cap \phi = \phi$.

XIII. Show that $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

XIV. If $A = \{a, b\}$ and $B = \{x, y\}$, then write the elements of the set $A \times B$ and $B \times A$.

Show that $A \times B \neq B \times A$.

Relations. We have been talking about relations like, on, similar, equal, congruent etc. This concept of relations need to be discussed in little detail. Relations between the sets of elements is an important tool in the hands of the mathematician. The operation of addition is the relation '+'. The pair of integers (2, 3) and the integer 5 has a relation. In elementary mathematics the examples of relations are :

equal ($=$) ; less than ($<$) ; greater than ($>$) ; congruent ($\cong$) ; similar ($\sim$) ; parallel ($\parallel$), perpendicular ($\perp$), etc. etc.

Supposing a famous theatre is having shows in the town and the price per ticket is Rs. 3. Now a ninth class of a school decides to see the performance together. So every member of the class will have to pay Rs. 3. Supposing there are 40 students in the class. If only one student agrees to visit the show then the cost will be Rs. 3, if two students agree then the cost will be Rs. 6 etc. In this way, we can construct the following set of ordered pairs :

$F=\{(1, 3), (2, 6), (3, 9)\ldots\ldots\ldots(40,120)\}$. This set gives us the relation between the number of boys who decide to visit the show and the price they have to pay. Now the first element of each pair has been taken from the set $D=\{1, 2, 3,\ldots\ldots 40\}$. The set D is called the domain of F. The second component of each ordered pair has been taken for the set $R=\{3, 6, 9\ldots\ldots 120\}$. The set R is called the range of F.

The formula $y=3x$ will express the relation between the number of students who decide to visit the show and the price they have to pay. We can write

$$F \text{ as } F=\{(x, y) \mid x\in D \text{ and } y=3x\}$$

If a, b are any two elements of the set then a, b can have some relation between them. For example,

$$a>b,\ a=b,\ a<b,\ a\sim b,\ a\cong b \text{ etc. etc.}$$

In general, we can write $a \mathrel{R} b$ where R stands for any relation. There can be a relation between a and b if

$$a=b \text{ or } a \mathrel{R} b.$$

Properties of Relations

Now some important relations have three especially important properties :

Property 1. If $x \mathrel{R} x$ for all elements of domain of R, then R is said to be reflexive. All the relations are not reflexive. $>$ is not reflexive as x cannot be greater than x or $x \ngtr x$.

But $x=x$; hence equality is a reflexive relation. A triangle is congruent to itself. Hence congruence is a reflexive relation. A triangle is similar to itself, hence similarity is a reflexive relation. A line cannot be perpendicular to itself, hence perpendicularity is not a reflexive relation. 'Is the sister of' or 'is the brother of' are not reflexive relations but 'as heavy as' or 'as long as' are reflexive relations. $\subset$ or $\supset$ are also reflexive relations.

Property 2. If x R y and y R x for all x in the domain of R and all y in the range of R then the relation is said to be symmetric. Now if $x=y$ then $y=x$, therefore equality is a symmetric relation but if $x>y$ then $y \ngtr x$; so greater than is not a symmetric relation. Similarly, less than is not a symmetric relation. But congruence and similarity are symmetric relations as if a $\triangle$ABC is congruent to $\triangle$ DEF then $\triangle$ DEF is congruent to triangle ABC. Similarly, if a $\triangle$ ABC is similar to $\triangle$DEF then $\triangle$DEF is similar to $\triangle$ABC. Similarly, parallelism and perpendicularity are symmetric relations. $\subset$ or $\supset$ are not symmetric.

Property 3. If xRy and yRz then xRz. In such a case R is said to be transitive relation.

Now if $a=b$ and $b=c$ then $a=c$, therefore equality is a transitive relation. Also if $a>b$ and $b>c$, then $a>c$, therefore greater than is a transitive relation. Similarly, less than, is a transitive relation.

Congruence, similarity, parallelism, $\supset$, $\subset$ are transitive relations. Perpendicularity is not a transitive relation, since if there are three lines l_1, l_2, and l_3 and $l_1 \perp l_2$ and $l_2 \perp l_3$ then l_1 will not be perpendicular to l_3.

Elementary Logic. In geometry or in any mathematical system, we apply laws of logic to draw inferences from statements and it is essential to know what a statement is.

A *statement* is a sentence which is either true or false. A statement is a declarative sentence. Of course, every sentence is not a statement. The sentence "give me the pen" is not a statement. Even every declarative sentence is not a statement. The declarative statement "I am thirsty" is not a statement as its truth or falsity will depend on the context or circumstances. A statement is generally a complex sentence in which ideas or words have been connected by the words such as 'and', 'or', 'if', 'then', 'not', 'every', 'any', 'all', 'some', etc.

Propositions. A proposition is a statement which is either true or false but not both at the same time. In Geometry, a statement is true if it is consistent with the postulates, and does not contradict any of the postulates or the theorems (or propositions) that have been derived earlier.

If the statement contradicts any of the postulates or the theorems that have been derived from them, it is false. Following connectives are very often used in propositions.

Connectives

Connective word	Symbol	Name
(1) 'If...then'	$\rightarrow$	Implication or conditional
(2) 'If and only if' or 'iff'	$\Leftrightarrow$	bi-conditional
(3) 'and'	$\wedge$	conjunction
(4) 'or'	$\vee$	disjunction
(5) 'not'	$\sim$	negation or denial

The first four connectives connect two statements to form a composite statement. In Geometry, we will be very often dealing with the statements containing the connectives "If...then" and "If and only if". 'If I live in Delhi then I live in India' is a statement of 'If...then' form. The if-clause gives us the hypothesis or what is given and the then-clause gives us the conclusion or what is to be proved. The use of the word 'then' is not very necessary. We could have written this very statement as 'If I live in Delhi I live in India' or 'I live in India if I live in Delhi'. Of course in both these statements, the if-clause gives us the hypothesis and the other clause gives us the conclusion. Another statement of this form would be "If two lines intersect then the measure of opposite angles is equal." Another one would be "If two lines are parallel then they do not intersect." In all these statements, the if-clause gives the hypothesis.

'If and only if' (or Iff) statements. Some examples of the statements of this form are :—

(1) A triangle is equilateral if and only if the measures of its three sides are equal.

(3) Two lines are parallel if and only if they do not intersect.

(3) Two lines are perpendicular if and only if they intersect at right angles.

In the if...then statement, if 'p' is the if-clause and 'q' is the 'then-clause', then the statement will be symbolically represented by $p \rightarrow q$ or p implies q which in other words means 'If p is true then q is true'. If it is true that I live in Delhi then it is true that I live in India is an illustration for it. Now if it is true that I live in India then it may not be true that I live in Delhi.

Hence q does not imply p. Now consider another if...then statement. 'If a triangle is equilateral then the measures of its three sides are equal ; in other words, if p then q. But in this case if q then p is also true as if the measure of three sides of a triangle is equal then the triangle is equilateral. So in this case $p \rightarrow q$ and $q \rightarrow p$. So $p \Leftrightarrow q$ or the statement is of 'if and only if' form.

If a statement has been obtained by connecting two propositions p, q with 'and', we get the statements 'p and q' or $p \wedge q$.

Now $p \wedge q$ will be true if both p and q are true.

If either one of p, q or both p and q are false then $p \wedge q$ is also false. Let p be the statement :—

A line l_1 is parallel to a line l_2 or $l_1 \parallel l_2$.

Let q be the statement :—

A line l_2 is parallel to the line l_3 or $l_2 \parallel l_3$.

Then the statement $p \wedge q$ will be $l_1 \parallel l_2$ and $l_2 \parallel l_3$. $p \wedge q$ will be true if both p and q are true. If we obtain a statement by the use of connective 'or' then the statement will be true if either one of p or q is true. Taking this very example of parallel lines, the statement p or q or $p \vee q$ will be $l_1 \parallel l_2$ or $l_2 \parallel l_3$ and $p \vee q$ will be true if either p is true or q is true or even both p and q are true. It may be noted that disjunction is used in inclusive sense and $p \vee q$ means either p or q or both.

A statement and its negation. The negation of a statement or a proposition p is the proposition, not—p which is symbolically represented as $\sim p$. $\sim p$ means p is false. If p is a statement 'the base angles of a triangle are equal', then $\sim p$ is the statement 'the base angles of a triangle are not equal'. It is clear that p and $\sim p$ (or $p \wedge \sim p$) cannot be true. In other words, both p and $\sim p$ cannot be true at the same time or $\sim (p \wedge \sim p)$ is true.

Syllogisms. While discussing deductive reasoning we discussed the implications of a statement.

Example :

General statement. If Arithmetic is easy for Kamla, then Algebra is easy for Kamla.

Specific statement. Arithmetic is easy for Kamla.

Conclusion. Algebra is easy for Kamla.

Our general statement consists of two parts : (1) If Arithmetic is easy for Kamla. Let us call it p. (2) Then Algebra is easy for Kamla may be called q. We can then say that if p then q or p implies q or $p \rightarrow q$. Now the statement p is called antecedent and the statement q is called consequent. This statement only tells us that if p is true then q is also true. We don't know whether both p and q are true or either one of these is true and the other is false or both are false. Now our specific statement tells us that p is true. Then we draw the inference that q is true. We can represent this inference in the following form :

1. $p \rightarrow q$
2. $\underline{p \qquad}$
3. $\therefore\ q$

In words we can say that p implies q and p is true, therefore q is true. This three step arrangement is called a syllogism. Steps 1 and 2 are called premises and step 3 is called inference or conclusion. The steps 1 and 2 of the arrangement given above could have been written in any order as pointed out earlier. This very syllogism could have also been written as follows :

1. p
2. $\underline{p \rightarrow q}$
3. $\therefore\ q$

Validity and Truth. As pointed out earlier, the inference drawn from the given conditional statement (If-then statement) could be valid even though either one or both of p, q may be false. It may therefore be understood that truth is a property of a statement but not a property of inference while validity is a property of inference but not of a statement. We cannot speak of a statement being valid or invalid or an inference being true or false. Validity of method of drawing the inference depends on the structure of the argument and is not dependent on the truth or falsity of premises or conclusions. Valid reasoning can lead to true or false conclusions. Also invalid reasoning can lead us to true or false conclusions.

General Statement. If Arithmetic is easy for Kamla, then Algebra is easy for Kamla.

Specific Statement. Algebra is easy for Kamla.

Conclusion. Arithmetic is easy for Kamla. In this case our three-step arrangement would be

1. $p \rightarrow q$
2. $\underline{q \qquad}$
3. $\therefore\ p$

Now this reasoning is invalid or the inference is invalid and we do not know whether conclusions drawn are true or false as we do not know if Algebra is easy for Kamla then Arithmetic will also be easy for Kamla. To help us understand the invalid form of reasoning, we have to understand the ideas of converse, opposite and contrapositive of a proposition. Let us consider the following proposition : If two triangles are congruent then these are also similar. In other words $p \rightarrow q$. Its converse would be : If two triangles are similar then these are also congruent or $q \rightarrow p$. The opposite of this would be : If two triangles are not congruent, then these are not similar or $\sim p \rightarrow \sim q$. The contrapositive of the given proposition would be : If two triangles are not similar then these are not congruent or $\sim q \rightarrow \sim p$. We can put these in a tabular form as follows :

Proposition : $p \rightarrow q$ (If p, then q).
Opposite : $\sim p \rightarrow \sim q$ (If not p, then not q).
Converse : $q \rightarrow p$ (If q, then p).
Contrapositive : $\sim q \rightarrow \sim p$ (If not q, then not p).

Following are some of the examples showing that a statement could be true or false and in each of these cases an inference can be valid or invalid.

Example 1. If the numeral for a whole number ends in '5' then the number is exactly divisible by 5. 735 is a whole number that ends in five, therefore 735 is divisible by five.

Example 2. If the unit's digit of a whole number is 5, then the number is exactly divisible by 6. If a number is exactly divisible by 6, then it is exactly divisible by 5. 735 is a whole number whose unit's digit is 5. Hence 735 is divisible by 5.

Example 3. If a whole number is exactly divisible by 2, then it is exactly divisible by 3. If it is exactly divisible by 3 then it is exactly divisible by 5. Hence if a number is exactly divisible by 2, it is exactly divisible by 5.

Example 4. If the measure of each angle of a triangle is 60 then the three sides of the triangle have equal measure. The three sides of a $\triangle$ABC have equal measure. Hence the measure of each angle of a triangle ABC is 60.

Example 5. If the earth is stationary, the moon is stationary. If the sun is stationary then the moon is stationary. So if the earth is stationary, the sun is stationary.

Analysis of validity or otherwise of the inferences and truth or otherwise of the conclusions is tabulated below :

Example	*First Reason*	*Second Reason*	*Inference*	*Conclusion*
1	True	True	Valid	True
2	False	False	Valid	True
3	False	False	Valid	False
4	True	True	Invalid	True
5	False	False	Invalid	False

From this table, we see that inference and its conclusion can both be valid and true (Exs. 1 and 2), valid and false (Ex. 3), invalid and true (Ex. 4) and invalid and false (Ex. 5).

Fundamental Laws of Logic

1. Law of Contradiction. For any proposition p, p is either true or false. It cannot be both true and false at the same time. In symbols, we can say that $p \wedge (\sim p)$ is false or $\sim [p \wedge (\sim p)]$ is true.

II. Law of Excluded Middle. For any proposition, there are only two possibilities (1) either it is true (2) or it is false. The statement that a proposition can neither be true nor false at the same time is not true. In other words, $p \vee (\sim p)$ is true. If a proposition is not true then it must be false. If a proposition is not false then it must be true.

What is a Proof? In our daily lives we very commonly use the word 'proof'. Some illustrations of it are :

(1) 'Out of the hundred people interviewed, 80 people were in favour of using a particular brand of tooth paste which is a proof that it is the most popular brand.'

(2) 'Let us fight a bout. It will prove as to which of us is the best fighter.'

(3) 'Sir, I am right. Twenty-nine minus fourteen equals fifteen as fifteen plus fourteen equals twenty-nine.'

(4) 'There is no need of any further proof. I saw it happen with my own eyes.'

(5) 'It is correct as my teacher told it so. There is no need of any proof for it.'

(6) 'It is true because my dad says like this. Where is the need of any proof ?'

In Mathematics, the word proof has very special and precise meaning. Proof in Mathematics consists of establishing a conclusion through a chain of conclusions drawn through valid inference. Each conclusion is justified by referring to the axioms or postulates that we have accepted, undefined terms, definitions, previously proved propositions or a combination of one or more of these. We also make use of assumptions and laws of logic and proved propositions of logic (we will take some of these later). None of the illustrations given above, excepting example 3 is an example of a valid proof as in other cases it is dependent upon induction or some authority like teacher or dad. Mathematical proof is independent of observation and experimentation. Observation and experimentation only suggest the axioms or postulates which we take as our unproved propositions.

Fundamental Rule of Inference. If we accept that $p \rightarrow q$ and p is true, then its logical implication is that q is also true. This pattern of reasoning is known as Fundamental Rule of Inference. We extend this reasoning further and say that if $p \rightarrow q$ and $q \rightarrow r$ then $p \rightarrow r$ and if p is true then r is also true. We can generalise it further and state that if $p \rightarrow q$, $q \rightarrow r$, $r \rightarrow s$, $s \rightarrow t$, $t \rightarrow u$ etc. etc., then $p \rightarrow u$.

As already stated, following forms of reasoning are invalid :

(1) If $p \rightarrow q$ and q is true then p is true. An example is : 'If two angles are right angles then they are equal in measure (or $p \rightarrow q$).' Conclusion : Two angles are equal in measure (or q) hence each angle is a right angle (or p). This inference is invalid.

(2) Another form of invalid reasoning is to accept that $p \rightarrow q$ and if $\sim p$ is true then $\sim q$ is true. In other words if $p \rightarrow q$ then $\sim p \rightarrow \sim q$ or to take that a proposition and its opposite are equivalent. An example is : If two angles are right angles then they are equal (or $p \rightarrow q$). Conclusion : Two angles are not right angles (or $\sim p$). Hence two angles are not equal (or $\sim q$).

Now let us consider the case when if $p \rightarrow q$ is true then will $\sim q \rightarrow \sim p$? Let the given proposition will be : 'If two angles are right angles then they are equal in measure (or $p \rightarrow q$). Its contrapositive would be 'If two angles are not equal in measure then both are not right angles.' Evidently, in this case if $p \rightarrow q$ then $\sim q \rightarrow \sim p$. Similarly if $\sim q \rightarrow \sim p$ is taken to be true then $p \rightarrow q$ is true. In the case of this example, therefore, $p \rightarrow q$ and $\sim q \rightarrow \sim p$ are logically equivalent statements or $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$.

We can show that in any case $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$. It is as follows :

1. $p \rightarrow q$ (We assume it to be true)
2. $\sim q$ (,, ,, ,, ,, ,,)
3. ∴ (?)

Obviously, $\sim q \rightarrow p$ or $\sim q \rightarrow \sim p$ are] the only two possibilities. Now assume that p is true then it implies that q is true and hence $\sim q$ is false and this contradicts the original assumption that $\sim q$ is true.

Hence the assumption that p is true has to be abandoned.

Hence $\sim q \rightarrow \sim p$. Therefore, $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$.

Also

(1) $\sim q \rightarrow \sim p$ (Assume it to be true)

(2) p (,, ,, ,, ,,)

(3) ∴ (?)

Once again we have to reject the possibility that $p \rightarrow \sim q$. Hence $p \rightarrow q$.

Another logical equivalence is double negation. For every proposition p, we have

$$[\sim(\sim p)] \leftrightarrow p$$

Suppose the proposition p is that sum of the angles of a triangle is two right angles, then $\sim p$ will be that sum of the angles of a triangle is not two right angles and $\sim(\sim p)$ would be that : "it is false that sum of angles of a triangle is not two right angles" and which amounts to saying that the sum of the angles of a triangle is two right angles.

We can establish the logical equivalence of $p \rightarrow q$ and $\sim q \rightarrow \sim p$ with the help of what are known as Truth tables. Before we are in a position to do so, we have to establish the truth and falsity of compound sentences like $p \wedge q$, $p \vee q$, $p \rightarrow q$ and for $\sim p$.

Truth Table for p and $\sim p$. If p is any proposition then if p is true then $\sim p$ is false and if p is false then $\sim p$ is true. We can put it in the following tabular form :—

When p is	Then $\sim p$ is
T	F
F	T

Here T stands for true and F stands for false. For example, let p be : "The measure of an angle A is 90", then $\sim p$ will be : "The measure of angle A is not 90".

Truth Table for $p \wedge q$. If p, q are any two statements then p is either true or false. Similarly, q is either true or false. Now $p \wedge q$ (p and q) will be true if and only if both p and q are separately true.

Let p be : 'Angle A is a right angle' and let q be 'Angle B is an acute angle' then $p \wedge q$ will be : 'Angle A is a right angle and angle B is an acute angle' and $p \wedge q$ will be true only when each of p, q is true ; otherwise it will be false. Hence its truth table is as follows :

When p is	And q is	Then $p \wedge q$ is
T	T	T
T	F	F
F	T	F
F	F	F

Truth Table for $p \vee q$. Now $p \vee q$ (p or q) will be true if either one of p or q is true. The statement : 'Angle A is a right angle or angle B is an acute angle' will be true if either one of p or q is true. Hence its truth table is :

When p is	And q is	Then $p \vee q$ is
T	T	T
T	F	T
F	T	T
F	F	F

Truth Table for $p\to q$. $p\to q$ means if p is true then q is true. In other words, if p is true then $\sim q$ is false or $p\wedge\sim q$ is false or $\sim(p\wedge\sim q)$ is true. Hence $p\to q$ has the same truth value as $\sim(p\wedge\sim q)$. Hence to build up the truth table for $p\to q$, we do so for $\sim(p\wedge\sim q)$.

When p is	When q is	When $\sim q$ is	And $p\wedge\sim q$ is	Then $\sim(p\wedge\sim q)$ is
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Now as $\sim(p\wedge\sim q)$ has the same truth value as $p\to q$, therefore, we get the following truth table for $p\to q$:

When p is	When q is	Then $p\to q$ is
T	T	T
T	F	F
F	T	T
F	F	T

Truth Table for $p\leftrightarrow q$ $(p\to q\wedge q\to p)$.

When p is	When q is	$p\to q$	$q\to p$	$p\leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Truth Table for $p\to q$ and $\sim q\to\sim p$.

When p is	When q is	Then $p\to q$ is	When $\sim q$ is	When $\sim p$ is	Then $\sim q\to\sim h$ is
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

This table clearly shows that $p\to q$ and $\sim q\to\sim p$ has the same truth value *i.e.* when $p\to q$ is true then $\sim q\to\sim p$ is also true ; when $p\to q$ is false then $\sim q\to\sim p$ is also false.

Strategy of Proofs in Geometry

In Geometry, we are to prove propositions which are generally conditional statements. Suppose the proposition is : 'If the measure of two sides of a triangle is equal then the measure of angles opposite these sides is also equal.' We assume the antecedent (if-clause) and then by arguing we prove that the consequent (then-clause) is true. The validity of the arguments given should be consistent with the laws of logic and propositions of logic. If argument is successful then we say that if p is true then q is also true, or $p \rightarrow q$. In general, we will find statements r, s, t, etc......... such that

$$(p \rightarrow r) \wedge (r \rightarrow s) \wedge (s \rightarrow t) \ldots\ldots \rightarrow (p \rightarrow q).$$

This type of strategy for proof is also known as 'direct proof' or 'modus poneus'.

Indirect Proof. Sometimes it is easier to prove a proposition easily by an 'indirect' scheme of proof. Essentially there are three types of indirect reasoning which are as follows :

(1) Proof by Reductio ad Absurdum.

(2) Proof by proving the contrapositive.

(3) Proof by elimination of all possibilities but one.

Reductio ad Absurdúm. In this type of proof we accept the contradictory statement of the given proposition and then prove that as a consequence of this we arrive at some inconsistent proposition or some 'absurd result'. If p is any proposition then we accept $\sim p$ and arrive at some contradiction and hence conclude that $\sim p$ is false or $\sim(\sim p)$ is true and then $\sim(\sim p) \rightarrow p$. This method has been very commonly used by mathematicians.

Example. A class consists of 40 students and the school has 39 copies of a particular book. If the whole class is to read that book simultaneously then two students will have to share the book. Assume the conclusion to be false. This implies that each student gets a book. This implies that class is having 40 books which contradicts the statement that the school has 39 books.

By proving the Contrapositive. In this type of proof we make use of the logical equivalence $(p \rightarrow q) \rightarrow (\sim q \rightarrow \sim p)$. We prove the contrapositive of the proposition and then assert that $p \rightarrow q$ is true. Suppose we want to prove the proposition that 'if two lines are cut by a transversal making the alternate interior angles equal in measure then the lines are parallel'. We write its contrapositive : "If two lines are not parallel then the alternate interior angles formed by any transversal are not equal in measure." We prove the contrapositive and then assert that the original proposition is proved.

By Elimination of All Other Possibilities. This type of reasoning is the simplest and very commonly used. Suppose we are confronted with a situation in which we want to prove that the lines l_1 and l_2 meet at a point. We will assert that either

(1) l_1 and l_2 are parallel ; or

(2) l_1 and l_2 are skew ; or

(3) l_1 and l_2 meet at a point.

We assert that these are the only three possibilities in which two lines in space are going to behave. Then we prove that the first two possibilities lead to contradictions. Often it is done by *Reductio ad absurdum*. Then we assert that the third possibility is true. If $p_1, p_2, p_3 \ldots\ldots p_n$, in general are the possibilities in a situation, we prove that $p_1, p_2 \ldots\ldots p_{n-1}$ are false or $\sim p_1, \sim p_2 \ldots\ldots \sim p_{n-1}$ are true and hence p_n is true.

$$[(p_1 \vee p_2 \vee p_3 \ldots\ldots p_{n-1} \vee p_n) \wedge (\sim p_1) \wedge (\sim p_2) \ldots\ldots \wedge (\sim p_{n-1})] \rightarrow p_n.$$

We must take care of two very important points in this type of proof :

(1) We have listed all the n possibilities and none has been left out.

(2) The $(n-1)$ possibilities have been proved to be false by a process of valid reasoning.

Summarising, we state that in this type of proof, we start with the conclusion and having formulated all the possible alternatives, we proceed backwards and arrive at untenable assumptions in the case of all other alternatives excepting one that we want to prove.

What is a Theorem ? A theorem is a statement that is to be proved. When we have proved it then it becomes a general statement for other theorems. A theorem and an assumption (postulate) are both statements. They differ only in this respect that an assumption is accepted to be true without any proof while a theorem is accepted to be true only when we have proved it. The proof of a theorem will have the following structure :—

(1) A statement of the theorem.

(2) A statement of hypothesis or what is given.

(3) A statement of what is to be proved.

(4) The figure that helps us to develop a logical proof. We do not allow our proofs to depend on any characteristics of the figure.

(5) A logical, step by step, proof based on deductive reasoning.

Proofs to be Arranged Column-wise. It is very convenient and helpful to arrange the proof of any problem involving deductive reasoning in column form. We will have two columns—a 'statement' column and a 'reason' column. We will start with hypothesis in the statement column. Every statement that we will make will be supported by valid reason. We will go on deducing further specific statements till we arrive at the conclusion. This in general will be our scheme for writing proofs when we employ the strategy of 'direct proof'. However, when we employ any one of the types of 'indirect proof' this scheme will be a little bit varied.

Arithmetic—An Axiomatic System

We have already stated that an axiomatic system has to be 'consistent'. The 'consistency' and 'independence' of axioms has to be ensured. A set of axioms will be consistent if it is not possible to deduce both the statement and its negation from the set of axioms by recognised customary rules of inference. In fact one can never deduce both the statements p and $\sim p$. All that the mathematicians have achieved is to reduce the consistency of Geometry to the consistency of Arithmetic. In other words, any inconsistency in the set of axioms of a particular geometry will imply inconsistency in the structure of elementary Arithmetic. Over four thousand years' experience has convinced the mathematicians that Arithmetic is a system which is consistent. The consistency of the system—the Euclidean geometry—that we are planning to build up in this book will depend on consistency of Arithmetic as a system.

Real number system is the core of Arithmetic and plays central role in Geometry. In real number system, the real numbers are the undefined terms, the operations of addition $(+)$ and multiplication $(\times)$ are the undefined relations. The real numbers include the following :

(1) Set of Natural numbers $\{1, 2, 3...\}$.

(2) The integer 0.

(3) The negative integers $\{-1, -2, -3...\}$.

(4) All positive and negative fractions whose denominator is different from zero.

(5) Irrational numbers like $\sqrt{2}$, $\sqrt{3}$ etc. We represent this mathematical structure or system by $[R, +, \times]$ where R stands for real numbers. It may be remembered that R, $+$, $\times$ are our undefined terms.

The axioms of this system are as follows :

A_1 : *Closure Property of Addition.* For all a, b belonging to R ; $a+b$ also belongs to R. Symbolically,

If $a, b \in R$ then $a+b \in R$. $2, 3 \in R$, therefore $2+3 \in R$.

A_2 : *Associative Property of Addition.* For all a, b, c belonging to R, $(a+b)+c=a+(b+c)$. $(\frac{1}{3}+\frac{2}{5})+\frac{5}{11}=\frac{1}{3}+(\frac{2}{5}+\frac{5}{11})$.

A_3 : *Identity Element for Addition.* There exists exactly one element of R which is denoted by 0 and which has a property that $a+0=a=0+a$ for all $a \in \mathrm{R}$. $\sqrt{2}+0=\sqrt{2}=0+\sqrt{2}$.

A_4 : *Existence of Inverse for Addition.* For all $a \in \mathrm{R}$, there exists one element, $-a$, in R, such that

$$a+-a=0=-a+a$$

$$\tfrac{7}{11}+-\tfrac{7}{11}=0=-\tfrac{7}{11}+\tfrac{7}{11} \text{ or}$$

$$-5+5=0=5+-5$$

A_5 : *Commutative Property for Addition.* For all $a, b \in \mathrm{R}$

$$a+b=b+a$$

$$\sqrt{3}+7=7+\sqrt{3}$$

or
$$\frac{-5}{3}+-\sqrt{7}=-\sqrt{7}+\frac{-5}{3}$$

Along with these five axioms for addition we have five similar axioms for multiplication and there is one axiom combining addition and multiplication.

These are as follows :

M_1 : *Closure Property in Multiplication.* For all $a, b \in \mathrm{R}$

$$a \times b \in \mathrm{R}.$$

M_2 : *Associative Property in Multiplication.* For all $a, b, c \in \mathrm{R}$

$$(a \times b) \times c=a \times (b \times c)$$

M_3 : *Identity Element in Multiplication.* There exists one element of R denoted by 1 such that :

$$a \times 1=a=1 \times a \text{ for all } a \in \mathrm{R}.$$

M_4 : *Existence of Inverse in Multiplication.* For all $a \in \mathrm{R}$ when $a \neq 0$, there exists exactly one element $\frac{1}{a}$ in R called the reciprocal of a such that $a \times \frac{1}{a}=1=\frac{1}{a} \times a$.

M_5 : *Commutative Property of Multiplication.* For all $a, b \in \mathrm{R}$

$$a \times b=b \times a.$$

AM_1 : *The Distributive Law.* For all $a, b, c \in \mathrm{R}$

$$a \times (b+c)=a \times b+a \times c=(b+c) \times a.$$

The distributive law combines addition and multiplication.

Besides these axioms, we need to make the following statements :

E_1 : If $a=b$ and $c=d$, then $a+c=b+d$ $(a, b \in \mathrm{R})$.

This is same as Euclid's Axiom 'If equals be added to equals, the sums are equals.'

E_2 : If $a=b$ and $c=d$, then $ac=bd$ $(a, b \in R)$.

This is a similar relation for the operation of multiplication.

We also need to mention that the relation of equality possesses all the three properties of relations about which we have mentioned earlier. We state those again.

Property 1. Equality is a reflexive relation, *i.e.*, for all $x \in R$; $x=x$.

Property 2. Equality is a symmetric relation, *i.e.*, for all $a, b \in R$ if $a=b$, then $b=a$

Property 3. Equality is a transitive relation, *i.e.*, for all $a, b, c \in R$ if $a=b$, $b=c$, then $a=c$

Besides properties of 'equality relation', we state following laws about 'inequality relation'.

O_1 : *Trichotomy Law.* For all $a, b \in R$, either $a<b$ or $a=b$ or $b<a$. In other words, one and only one of these relations holds.

O_2 : *Inequality is a Transitive Relation.* For all $a, b, c \in R$ if $a<b$ and $b<c$, then $a<c$. Of course, inequality is not a reflexive or a symmetric relation.

AO_1 : For all $a, b, c \in R$ if $a<b$, then $a+c<b+c$.

This is similar to E_1.

MO_1 : For all $a, b \in R$ if $a>0$ and $b>0$, then $ab>0$.

This is similar to E_2.

We also state the following statements which are satisfied by real number system.

The Archimedean Postulate. If a, b are any two positive real numbers or any magnitudes in geometry then there exist positive integers m and n such that

$$ma>b \text{ and } nb>a.$$

The Euclidean completeness postulate :

Every positive real number has a positive square root.

Subtraction. In real number system, we also deal with another operation *viz.* subtraction which is inverse of addition. Axiom A_4 helps us to state that $a-b=a+(-b)$. In other words, to subtract b from a, add $(-b)$ to a.

Division. It is an operation inverse of multiplication. Axiom

M_4 helps us to state that $a \div b = a \cdot \frac{1}{b}$. In other words to divide a by b, multiply a by reciprocal of b.

With the help of these definitions, axioms, statements, order-relations, we prove below some theorems. The pattern of proof that we will follow in proving these will be same as that we follow in proving the theorems in geometry.

Theorem 1. For every $a \in R$, $a \times 0 = 0$.

Given : $a \in R$.

To prove that : $a \times 0 = 0$.

Proof.

Statement	*Reason*
1. $1+0=1$	1. Axiom A_3
2. $a(1+0)=a\times 1$	2. Statement E_2
3. $a\times 1 + a\times 0 = a\times 1$	3. Distributive law Axiom AM_1
4. $a + a\times 0 = a$	4. Axiom M_3
5. $a + a\times 0 + -a = a + -a$	5. Statement E_1
6. $a + -a + a\times 0 = a + -a$	6. Axiom A_5
7. $(a + -a) + a\times 0 = (a + -a)$	7. Axiom A_2
8. $0 + a\times 0 = 0$	8. Axiom A_4
9. $a\times 0 = 0$	9. Axiom A_3.

Theorem 2. If a, b are any two real numbers such that $ab=0$, then $a=0$ or $b=0$.

Given : $a, b \in R$ and $ab=0$.

To prove that : Either $a=0$ or $b=0$.

Proof. If we consider a ; then either $a=0$ or $a \neq 0$ (Indirect method of Proof). If $a \neq 0$, then $\frac{1}{a}$, the reciprocal of a exists and $\frac{1}{a} \in R$. Then

Statement	*Reason*
1. $ab=0$	1. Given
2. $\frac{1}{a}$ exists	2. $a \neq 0$ as has been supposed

3. $\frac{1}{a}\times(ab)=\frac{1}{a}\times(0)$	3. Statement E_2
4. $\frac{1}{a}(ab)=0$	4. Theorem 1
5. $\left(\frac{1}{a}\times a\right).\,b=0$	5. M_2
6. $1\,.\,b=0$	6. M_4
7. $b=0$	7. M_3.

So either $a=0$ or $a\neq 0$ but when $a\neq 0$ then $b=0$.

Hence either $a=0$ or $b=0$.

Theorem 3. For every $a\in R$, $(-1)\times a=-a$.

Given : $a\in R$.

To prove that : $(-1)\times a=-a$.

Proof.

Statement	*Reason*
1. $1+(-1)=0$	1. A_4
2. $a\times[1+(-1)]=a\times 0$	2. E_2
3. $a\times 1+a\times(-1)=0$	3. AM_1, Theorem 1
4. $a+a\times(-1)=0$	4. M_3
5. $(-a)+a+a\times(-1)=(-a)+0$	5. E_1
6. $[(-a)+a]+a\times(-1)=(-a)+0$	6. A_2
7. $0+a\times(-1)=-a+0$	7. A_4
8. $a\times(-1)=-a$	8. A_3
9. $(-1)\times a=-a$	9. M_5.

Theorem 4. For every $a, b\in R$, $(-a)b=-(ab)$.

Given : $a, b\in R$.

To prove that : $(-a)b=-(ab)$.

Proof.

Statement	*Reason*
1. $-a=(-1)\times a$	1. Theorem 3
2. $(-a)b=(-1)\times a\times b$	2. E_2
3. $(-a)b=(-1)(ab)$	3. M_2
4. $(-a)b=-(ab)$	4. Theorem 3.

Theorem 5. For every $a \in R$, prove that $-(-a)=a$.

Given : $a \in R$.

To prove that : $-(-a)=a$.

Proof. $a \in R$, therefore $(-a) \in R$ exists.

Statement	*Reason*
1. $-(-a)+(-a)=0$	1. A_4
2. $-(-a)+(-a)+a=0+a$	2. E_1
3. $-(-a)+[(-a)+a]=[0+a]$	3. A_2
4. $-(-a)+0=0+a$	4. A_4
5. $-(-a)=a$	5. A_3.

Theorem 6. For every $a, b \in R$, $(-a)(-b)=ab$.

Given : $a, b \in R$.

To prove that : $(-a)(-b)=ab$.

Proof.

Statement	*Reason*
1. $(-a)(-b)=-[a(-b)]$	1. Theorem 4
2. $(-a)(-b)=-[(-b)(a)]$	2. M_5
3. $(-a)(-b)=-[-(ba)]$	3. Theorem 4
4. $(-a)(-b)=ba$	4. Theorem 5
5. $(-a)(-b)=ab$	5. M_5.

Theorem 7. Cancellation law of addition.

For all $a, b, c \in R$ if $a+b=a+c$ then $b=c$.

Given : (*i*) $a, b, c \in R$. (*ii*) $a+b=a+c$.

To prove that : $b=c$.

Proof.

Statement	*Reason*
1. $a+b=a+c$	1. Given
2. $-a+(a+b)=-a+(a+c)$	2. E_1
3. $(-a+a)+b=(-a+a)+c$	3. A_2
4. $0+b=0+c$	4. A_4
5. $b=c$	5. A_3.

Theorem 8. Cancellation law of multiplication.

For all $a, b, c \in R$ if $ab=ac$ and $a \neq 0$, then $b=c$.

Given : (*i*) $a, b, c \in \mathrm{R}$. (*ii*) $ab = ac$. (*iii*) $a \neq 0$.

To prove that : $b = c$.

Proof.

Statement	*Reason*
1. $\frac{1}{a}$, the reciprocal of a exists	1. $a \neq 0$
2. $ab = ac$	2. Given
3. $\frac{1}{a}(ab) = \frac{1}{a}(ac)$	3. E_2
4. $\left(\frac{1}{a} \cdot a\right)b = \left(\frac{1}{a} \cdot a\right)c$	4. M_2
5. $1b = 1c$	5. M_4
6. $b = c$	6. M_3

Theorem 9. For all $a, b, c, d \in \mathrm{R}$ and if $a < b$ and $c < d$, then $a + c < b + d$.

Given : (*i*) $a, b, c, d \in \mathrm{R}$. (*ii*) $a < b$ and $c < d$.

To prove that : $a + c < b + d$.

Proof.

Statement	*Reason*
1. $a < b$	1. Given
2. $a + c < b + c$	2. AO_1
3. $c < d$	3. Given
4. $b + c < b + d$	4. AO_1
5. $a + c < b + d$	5. Statement 2, 4 and O_2.

We have proved that any two inequalities of the same order can be added.

Theorem 10. If we multiply both sides of an inequality by the same positive number then the inequality is preserved. In other words for all $a, b, c \in \mathrm{R}$, (*i*) if $a < b$ and $c > 0$, then $ac < bc$, (*ii*) if $a > b$ and $c > 0$, then $ac > bc$.

1st Part. *Given* : (*i*) $a, b, c \in \mathrm{R}$. (*ii*) $a < b$ and $c > 0$.

To prove that : $ac < bc$.

Proof.

Statement	*Reason*
1. $b > a$	1. Given that $a < b$
2. $b + -a > a + -a$	2. AO_1

3. $b+-a>0$	3. A_4
4. $c(b+-a)>c\times 0$	4. Given that $c>0$ MO_1
5. $c.b+c(-a)>c\times 0$	5. AM_1
6. $c.b+c(-a)>0$	6. Theorem 1
7. $cb-ca>0$	7. Theorem 4
8. $bc-ac>0$	8. M_5
9. $bc-ac+ac>0+ac$	9. $A0_1$
10. $bc+0>0+ac$	10. A_4
11. $bc>ac$	11. A_3
12. $ac<bc$	12. Statement 11.

2nd Part. It can be done by a reasoning similar to 1st part.

Theorem 11. For all $a\in R$, if $a<0$, then $-a>0$.

Given : (*i*) $a\in R$. (*ii*) $a<0$.

To prove that : $-a>0$.

Proof.

Statement	*Reason*
1. $a<0$	1. Given
2. $-a+a<-a+0$	2. AO_1
3. $0<-a+0$	3. A_4
4. $0<-a$	4. A_3
5. $-a>0$	5. Statement 4.

Theorem 12. For all $a\in R$, if $a>0$ then $-a<0$.

Proof.

Statement	*Reason*
1. $a>0$	1. Given
2. $0<a$	2. Statement 1
3. $-a+0<-a+a$	3. AO_1
4. $-a+0<0$	4. A_4
5. $-a<0$	5. A_3.

Theorem 13. An inequality is reversed if its both sides are multiplied by a number less than 0. In other words if (*i*) $a>b$ and $c<0$, then $ac<bc$; (*ii*) $a<b$ and $c<0$, then $ac>bc$, where $a, b, c\in R$.

1st Part.

Given : (*i*) $a, b, c\in R$, (*ii*) $a>b$ and $c<0$.

To prove that : $ac<bc$.

Proof.

Statement	*Reason*
1. $a>b$	1. Given
2. $a+-b>b+-b$	2. Why ?
3. $a+-b>0$	3. Why ?
4. $c<0$	4. Given
5. $-c>0$	5. Theorem 11
6. $-c(a+-b)>-c\times 0$	6. Why ?
7. $(-c)a+(-c)(-b)>-c\times 0$	7. AM_1
8. $(-c)a+(-c)(-b)>0$	8. Theorem 1
9. $-ca+cb>0$	9. Theorem 4, 6
10. $-ac+bc>0$	10. M_5
11. $ac-ac+bc>ac+0$	11. Why ?
12. $0+bc>ac+0$	12. Why ?
13. $bc>ac$	13. Why ?
14. $ac<bc$	14. Why ?

2nd Part. This can be proved by a reasoning similar to 1st part.

Another concept which we deal with in Algebra and which we shall be dealing with in Geometry also is that of absolute value of a number. Absolute value of a number x is denoted by $|x|$. By absolute value we mean the magnitude of a number without its sign. More precisely it can be defined by the following two conditions :

(1) $|x|=x$ when $x\geqslant 0$

(2) $|x|=-x$ when $x<0$.

For example, the absolue value of $+5$ is 5 as $+5\geqslant 0$ and that of -5 is $-(-5)$ or 5 as $-5<0$. In other words, the absolute value of a postive number is the number itself while that of a negative number is the number when the negative sign has been removed.

Theorem 14. Absolute value of any number is never less than zero or in other words $|x|\geqslant 0$ for any x where $x\in R$.

Given : $x\in R$.

To prove that : $|x|\geqslant 0$.

Proof. Either (*i*) $x>0$ or (*ii*) $x=0$ or (*iii*) $x<0$. (Indirect proof—exhausting all the possibilities.)

1st Case. When $x>0$, then by the first condition of definition of absolute value $|x|=x$ and x is greater than zero as it is given to be positive.

2nd Case. When $x=0$, then $|x|=x=0$.

3rd Case. When $x<0$, then $|x|=-x$ and as $x<0$, hence $-x>0$ (Theorem 11).

So in every case the absolute value of a number is never less than zero.

Theorem 15. Absolute value of any real number is never less than the number. In other words for all $x \in R$,

$|x| \geqslant x$.

Proof. Either $x>0$ or $x=0$ or $x<0$.

1st Case. When $x \geqslant 0$ then $|x|=x$.

2nd Case. When $x<0$ then $|x|=-x$ and as $x<0$, hence $-x>0$ or $-x>0>x$. Hence $|x|>x$.

Hence combining the results of first and second case, we get

$$|x| \geqslant x \text{ for all } x \in R.$$

Theorem 16. The absolute value of any real number and the absolute value of its additive inverse are the same. In other words, $|x|=|-x|$ for all $x \in R$.

Proof. Either $x \geqslant 0$ or $x<0$.

1st Case. When $x \geqslant 0$, then $-x \leqslant 0$. (Why ?)

If $x \geqslant 0$ then $|x|=x$ and then

$-x \leqslant 0$ and hence $|-x|=-(-x)=x$. (Why ?)

Hence $|x|=|-x|$.

2nd Case. When $x<0$, then $-x>0$. (Why ?)

If $x<0$, then $|x|=-x$ and when $-x>0$

then $|-x|=-x$. (Why ?)

Hence $|x|=|-x|$ for all $x \in R$.

Theorem 17. Absolute value of product of two real numbers is the product of the absolute values of two numbers. In other words for all $x, y \in R$, $|xy|=|x| \cdot |y|$.

Proof. Either (*i*) $x \geqslant 0, y \geqslant 0$ or (*ii*) $x<0, y<0$ or (*iii*) when either one of x, y is positive and the other is negative.

1st Case. If $x \geqslant 0, y \geqslant 0$, then $xy \geqslant 0$ and hence $|xy|=xy$ and $|x|=x$ and $|y|=y$. Therefore, $|x| \cdot |y|=xy$.

Hence $|xy|=|x|\ |y|$.

2nd Case. If $x<0$ and $y<0$, then $xy>0$.

Hence $|x|=-x$, $|y|=-y$ and $|xy|=xy$.

Therefore, $|x| \cdot |y|=(-x)(-y)=xy$.

Hence $|xy| = |x| . |y|$.

3rd Case. When $x<0$, and $y>0$, then $|x|=-x$ and $|y|=y$ and as $xy<0$, (Theorem 13) therefore $|xy|=-xy$.

Now $|x| . |y|=(-x)(y)=-xy$.

Hence $|xy| = |x| . |y|$.

Of course, there are some cases when $x=0$ and $y<0$.

Or, $y=0$ and $x<0$ etc. Then $|xy| = |x| . |y|$ as both sides of this equality will be zero in this case.

Hence $|xy| = |x| . |y|$ for all $x, y \in R$.

Theorem 18. Absolute value of sum of two real numbers is either less than or equal to the sum of the absolute values of two numbers,

or $|x+y| \leqslant |x| + |y|$.

Proof. Either $x+y \geqslant 0$ or $x+y<0$.

1st Case. If $x+y \geqslant 0$, then $|x+y|=x+y$.

Now if $x+y \geqslant 0$ then either (*i*) $x>0, y>0$

or (*ii*) $x<0, y>0$ with $|y|>|x|$

or $x>0, y<0$ with $|x|>|y|$

or $x=0, y=0$ or $x=0, y>0$

or $y=0, x>0$ are the all possible cases.

Now when $x>0, y>0$ then $|x|=x$ and $|y|=y$.

Hence $|x|+|y|=x+y$ and then $|x+y|=|x|+|y|$ when $x<0, y>0$ provided $|y|>|x|$ then $|x|=-x$ and $|y|=y$ and $|x|+|y|=-x+y=y-x$.

Also $y-x>y$ as $x<0$.

Moreover, $y>y+x$ as $x<0$. Hence $y-x>yx+y$.

Hence $|x|+|y|>|x+y|$ (as $|x+y|=x+y$ in this case)

or $|x+y|<|x|+|y|$.

Similarly, we can prove that if $x>0, y<0$ and $|x|>|y|$ then $|x+y|<|x|+|y|$.

When $x=0, y=0$ then $x+y=0$.

Hence $|x+y|=0$ and $|x|=0$, $|y|=0$ and hence $|x|+|y|=0$.

Therefore, $|x+y|=|x|+|y|$

when $x=0$ and $y>0$, then $x+y>0$ and hence $|x+y|=y$ and $|x|=0$, $|y|=y$ and hence $|x|+|y|=y$.

Therefore, in this case $|x+y|=|x|+|y|$.

Similarly, when $x>0$ and $y=0$, then $|x+y|=|x|+|y|$.

2nd Case. If $x+y<0$, then either (*i*) $x<0$ and $y<0$ or (*ii*) $x>0$ and $y<0$, but $|y|>|x|$ or (*iii*) $y>0$ and $x<0$ but $|x|>|y|$ or (*iv*) $x=0$ and $y<0$ or (*v*) $y=0$ and $x<0$.

When $x<0$ and $y<0$, then $(x+y)<0$ and then $|x+y| =-(x+y)$ and $|x|=-x$ and $|y|=-y$ and hence $|x|+|y| =-x-y=-(x+y)$.

Hence $|x+y|=|x|+|y|$.

When $x>0$ and $y<0$ but $|y|>|x|$, then $(x+y)<0$ and $|x+y|=-(x+y)$; $|x|=x$ and $|y|=-y$.

Therefore, $|x|+|y|=x-y$

while $|x+y|=-(x+y)=-x-y$.

Now as $x>0$ and hence $-x<0$.

Therefore, $x-y>-x-y$ and hence $|x|+|y|>|x+y|$

or $|x+y|<|x|+|y|$.

Similarly, we can prove that when $y>0$ and $x<0$

but $|x|>|y|$ then $|x+y|<|x|+|y|$.

Now when $x=0$ and $y<0$, then $(x+y)<0$

and then $|x+y|=-(x+y)=-y$ and $|x|=0$, $|y|=-y$

and hence $|x|+|y|=-y$ and then $|x+y|=|x|+|y|$.

Hence for all $x, y \in \mathrm{R}$ $|x+y| \leqslant |x|+|y|$.

Problem Set

Prove that if $p, q, x, y \in \mathrm{R}$, then

1. $p(-q)=-pq$.
2. $q(-p)=-pq$.
3. $(-p)(-q)=pq$.
4. $(-q)(-p)=pq$.
5. $p(q+r)=rp+rq$.
6. $q(p-r)=pq-rq$.
7. $p-0=p$.
8. $p+p=2p$.
9. $q+q+q+q=4q$.
10. $(-q)+(-q)=-2q$.
11. $|x^2|=|x|^2$.
12. $|x^3|=|x|^3$.
13. $|x-y| \geqslant |x|-|y|$ and if $|x-y|=0$, then $|x|=|y|$.

14. $|x+y| \geqslant |x| - |y|$.

15. $\big|\, |x| - |y| \,\big| < |x| + |y|$.

16. If $a+b=c+d$ and $a>c$, then $b<d$.

Sol. 1. $a>c$. (Given.)

2. $a+(-c)>c+(-c)$. Why ?
3. $a+(-c)>0$. Why ?
4. $a-c>0$. Why ?
5. Let $a-c=m$ where $m>0$. Statement 4.
6. $a+b=c+d$. Given.
7. $a+b+(-b)=c+d+(-b)$. Why ?
8. $a=c+d-b$. Why ?
9. $a-c=d-b$. Why ?
10. $m=d-b$. Why ?
11. $d-b>0$. Why ?
12. $d>b$. Why ?
13. $b<d$.

Another Example of Axiomatic System

Suppose a set of friends $\{x, y, z......\}$ has gone for mountaineering and they form themselves into groups $\{G_1, G_2, G_3...\}$ for doing different odd jobs such that following Axioms hold :

A_1 : There are atleast two members in the given set.

A_2 : Every group consists of members of the set.

A_3 : If x, y are any two members of the set then there is one and only one group that contains both x and y.

A_4 : If G_1 is any group then there is atleast one member of the set that is not in G_1.

A_5 : If G_1 is a set and there is a member Z which is not in G_1 then there is one and only one group containing z whose other members are also different from members of G_1.

Before we proceed to prove any theorems, let us try to understand the implications of these Axioms.

Axiom 1. The set has atleast two members. It can have more than two members.

Axiom 2. The groups consist only members of the set.

Axiom 3. x, y any two members of the set come together only in one group.

Axiom 4. It eliminates the possibility of only one group or any one group consisting of all the members of the set.

Axiom 5. There will be two and only two groups such that members of one group will be altogether different from the members of the other group.

Theorem 1. Every member of the set is atleast in two different groups.

Proof.

Statement	*Reason*
1. x is a member of the set	1. A_1
2. x, y determine a group G_1	2. A_3
3. z is a member of the set that does not belong to G_1	3. A_4
4. x and z determine a group G_2	4. A_3
5. x belongs to two groups G_1 and G_2	5. Statement 2&4.

Theorem 2. Every group has atleast one member of the set. (This is to eliminate the possibility of a group having no members at all or a group being an empty set.)

Proof.

Statement	*Reason*
1. x is a member of the set	1. A_1
2. x belongs to groups G_1 and G_2	2. Theorem 1.

From now on we shall argue on the lines of indirect Proof. Suppose there exists a member G_3 with no groups at all. Then members of G_3 are altogether different from the members of G_1 and G_2 and this contradicts the Axiom 5. Hence the supposition that G_3 has no members at all has to be abandoned.

Theorem 3. Every group contains atleast two members.

We shall employ the technique of 'proof by contradiction'.

Proof. Let G_1 be a group ; then :

1. G_1 has atleast one member (Theorem 2).
2. Suppose G_1 has only one member, x.
3. x belongs to another group G_2 (Theorem 1).

4. G_2 must have another member y as otherwise it will not be different from G_1.

5. There is another member, z, which is not in G_2 (A_4).

6. If z belongs to G_1 then G_1 will have two members and we have nothing further to prove.

7. Suppose z is not a member of G_1.

8. There exists a group G_3 containing z whose members are altogether different from G_3 (A_5).

9. Also G_3 is altogether different from G_1 as G_1 only contains x and x cannot belong to G_3 as x belongs to G_2 and in that case G_2 and G_3 will not be altogether different. So G_3 is altogether different from G_1 and G_2 and this contradicts A_5. Hence the supposition that G_1 contains only one member has to be abandoned.

Theorem 4. There are atleast four members in the set.

Proof.

Statement	*Reason*
1. x and y are members of the set	1. A_1
2. x and y belong to G_1	2. A_3
3. z is a member that does not belong to G_1	3. A_4
4. There is a group G_2 containing z whose members are altogether different from G_1	4. A_5
5. G_2 has another member a	5. Theorem 3
6. So x, y, z and a are the four members that belong to the set	6. Statement 1, 3, 5
7. Set has atleast four members	7. Statement 6

Theorem 5. There exist atleast six different groups.

Statement	*Reason*
1. x, y are the members of the set	1. A_1
2. x, y belong to G_1	2. A_3
3. z is a member that does not belong to G_1	3. A_4
4. There is a group G_2 containing z whose members are altogether different from G_1	4. A_5
5. G_2 has another member 'a'	5. Theorem 3
6. x, y, z, a belong to the set	6. Statement 1, 3, 5
7. (x, z), (x, a), (y, z), (y, a) determine groups different from G_1 and G_2	7. A_3
8. There exist atleast six different groups	8. Statement 2, 4, 7 .

Now we give below a set of five Axioms which are same as we have stated above (How ?). The undefined terms are 'points' and 'lines'.

A_1 : There exist atleast two points.

A_2 : Every line is a collection of points.

A_3 : If x, y are any two points then there is one and only one line containing both x, y.

A_4 : If l_1 is any line then there exists a point not on l_1.

A_5 : If l_1 is a line and z is a point which is not on l_1 then there exists one and only one line through z which is parallel to l_1.

Now with the help of these axioms, the reader need to prove the following theorems :

Theorem 1. Every point is on two different lines.

Theorem 2. Every line contains atleast one point.

Theorem 3. Every line contains atleast two points.

Theorem 4. There exist atleast four distinct points.

Theorem 5. There exist atleast six distinct lines.

(*Hint.* These two sets of axioms are the same. Only 'members' have been replaced by 'points' and 'groups' have been replaced by 'lines'.)

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CHAPTER II

EUCLIDEAN GEOMETRY AS AN AXIOMATIC SYSTEM

With all this preparation we are now well set for developing Euclidean geometry as an Axiomatic system. Point, line, plane, space, set, element, number shall be our undefined terms. Of course, we will have some 'understanding' of these terms. The relation 'belong to' will be the undefined relation. We will assume the existence of Real number system as an axiomatic system and hence we will accept the undefined terms, definitions, Axioms, Theorems etc. of Real number system which is going to play the central role in the development of Euclidean geometry as an axiomatic system. We have developed Real number system as a quadroplet [R, +, X, <] which will be the core of the system that we are going to develop.

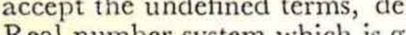


Fig. 12.

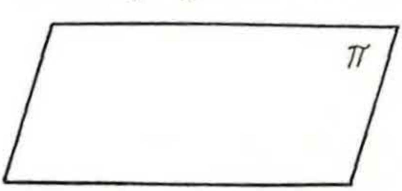


Fig. 13.

In geometry, we will start with space, S, which is a set of points. The term point will be taken as undefined. The definition of a point as having 'no length, width and thickness' is incrrect. A point will be something abstract. We can make geometrical representation of a point by a 'dot' made by a pencil. The dot will only be a geometrical representation of the 'point'. It will not be a 'point' as point is something abstract—undefined. Of course, 'point' will be element of set S. We will also take a subset of S—a line which will be a collection of points. The term 'line' will be taken as undefined. We can make the geometrical representation of the line as in Fig. 12. Our line is going to extend both ways. Of course this figure is geometrical representation of a line but it is not line itself. Line is something 'abstract' which exists only in our imagination. We are going to omit the adjective 'straight' as 'line' is something undefined but in our thinking and imagination the line is going to be 'straight' so far as it is going to exist on a plane paper or in our immediate ueighbourhood. But if our 'line' is going to extend into space constituting the universe we live in, then we do not know whether it is going to be straight or something different. We will also be taking another subset of S namely plane which will be collection of points. We will represent a plane geometrically by a parallelogram as in Fig. 13. Of course, our plane will be undefined. Then we are going to develop the system by using the relations, like, 'incidence'' 'on' or 'belong to', 'between', 'equal', 'congruent', 'similar', 'less than' 'parallel', 'perpendicular', etc.

To make it more clear it is stated that a mathematician's view point is different from that of a physicist. When a physicist uses the terms point, straight line, incidence etc. in a statement about physical objects, he has a definite meaning before him for these. A pin-point is a 'point' for a physicist. A tant string or the path of light rays in a homogeneous medium is a straight line. A mathematician does not assign any meaning to these terms—and develops the system with the help of laws of logic after making some assumptions or postulates. A mathematician is only worried about the 'consistency' of the system rather than its 'truth' or 'otherwise'. A physicist may give physical meaning to these terms and postulates and if he finds on the basis of experiments or systematic observation that the system that the mathematician has developed holds true in physical space then he may make use of it. The mathematician only gives the physicist an assurance that if the postulates that he has assumed are true in their physical interpretation then the theorems that he has developed are necessarily true as these are logically deducible from those postulates.

Although our points and lines are going to be undefined terms yet we shall make use of figures in our proofs and discussions to help us in formulating the proofs and to know about further relations that need to be explored. Of course we will not justify our proofs by basing those on characteristics of figures.

Besides assuming Axioms of real numbers, we will need twenty-two postulates. We may again make it clear that axioms are those assumptions that are common to all sciences and postulates are those assumptions which are peculiar to the particular science. The first group of postulates are those that are commonly called as 'postulates of incidence or connection'. These postulates express our intuitive ideas about how points, lines and planes are related. In these we use the relations as expressed by the words 'lies on' or 'contain' or 'determines' or 'intersects'.

The set of postulates that we are going to use is as follows :—

Postulates of Incidence :

Postulate 1. Every line is a set of points and contains atleast two distinct points.

Postulate 2. If A and B are any two distinct points then there is one and only one line containing both A and B.

Postulate 3. A plane is a set of points and contains atleast three non-collinear points.

Postulate 4. If A, B, C are any three non-collinear points then there is one and only one plane that contains all of these.

Postulate 5. If two points of a line lie in a plane then every point of the line lies in that plane.

Postulate 6. If two planes intersect (have a point in common), then their intersection is a line.

Postulate 7. Space contains atleast four points which are non-collinear and non-coplanar.

Distance Postulates :

Postulate 8 (*Distance postulate*). To every pair of different points in space there corresponds a unique number such that it is :

(1) 0, if the points of the pair are the same.

(2) positive, if the points of the pair are distinct.

(3) 1, if the points of the pair are at a unit distance.

Postulate 9 (*The Ruler Postulate*). The points of a line can be placed in a unique one-to-one correspondence with the set of all real numbers in such a way that :

(1) To every point of the line, there corresponds exactly one real number.

(2) To every real number, there corresponds exactly one point on the line.

(3) The measure of distance between any pair of points A and B on the line is equal to absolute value of the difference of numbers that correspond to A and B.

The Separation Postulates :

Postulate 10 (*The Plane Separation Postulate*). Given a line and a plane containing it, the points of the plane which do not belong to the line form two sets such that :

(1) Each set is convex.

(2) A segment having one of the end-points in one set and the other end-point in the second set, intersects the given line.

Postulate 11 (*The Space Separation Postulate*). Given a plane in space. The set of points of space which do not lie in the given plane form two sets such that :

(1) Each set is convex.

(2) Any segment joining a point in one set to a point in the other set, intersects the given plane.

The Angle Postulates :

Postulate 12 (*The Angle Measurement Postulate*). To every $\angle A$, there corresponds a real number between 0 and 180.

Postulate 13 (*The Angle Construction Postulate*). If $\overrightarrow{AB}$ is any ray on the edge of a half plane H, then for every real number r between zero and 180, there is exactly one ray $\overrightarrow{AK}$ with K in the half plane H such that $m \angle KAB = r$.

Postulate 14 (*The Angle Addition Postulate*). If P is a point in the interior of $\angle BAC$,

then $m \angle BAC = m \angle BAP + m \angle PAC$.

The Protractor Postulate :

If $\overrightarrow{VA}$ is any ray lying in the edge of a half plane H, there is one-to-one correspondence between the set of real numbers r where $0 \leqslant r \leqslant 180$ and the set of rays $\overrightarrow{VX}$ which lie in the union of H and its edge such that

(1) $\overrightarrow{VA}$ corresponds to the number 0.

(2) The ray opposite to $\overrightarrow{VA}$ corresponds to number 180.

(3) If X and Y are points belonging to H and which are not collinear with V and if x, y are the real numbers corresponding to $\overrightarrow{VX}$ and $\overrightarrow{VY}$ respectively then $m \angle XVY = | x - y |$.

Postulate 15 (*The Supplement Postulate*). If two angles form a linear pair, then they are supplementary.

The Congruence Postulate :

Postulate 16 (*S.A.S. Postulate*). If there exists one-to-one correspondence between two triangles (or between a triangle and itself) in which two sides and the angle *determined* by three sides of one triangle are congruent to corresponding parts of the second triangle, the correspondence is a congruence and the triangles are congruent.

The Parallel Postulate :

Postulate 17. Given a line and a point not on that line, there is only one line through the given point parallel to the given line.

Area Postulates :

Postulate 18. If S is any square region, then there exists a correspondence associating every polygonal region a unique positive number such that the given square region is assigned the number 1.

Postulate 19. If two triangles are congruent, then the triangular regions determined by them have the same area relative to a unit square.

Postulate 20. If a polygonal region R is the union of two polygonal regions R_1 and R_2 such that R_1 and R_2 intersect almost in a finite number of points and segments then the area of R relative to a unit square is the sum of the areas of R_1 and R_2 relative to the same unit square.

Postulate 21. If S is any square region whose sides are of a unit length then the area of any rectangular region R, relative to unit square S is the product of its two consecutive sides.

Co-ordinate Postulate :

Postulate 22. To every point in the plane, there corresponds a unique ordered pair of real numbers and to every ordered pair of real numbers, there corresponds a unique point in the plane.

As will be evident from Postulate 22, we intend to integrate analytical methods with the methods of pure geometry. We intend to develop elementary synthetic geometry and elementary analytical plane geometry and then to integrate them. We intend to develop the system under the following chapters :—

(1) Incidence postulates and theorems, (2) Distance and its measurement, (3) Planes, space and separation, (4) Angles and angle measurement, (5) Triangles and their congruence, (6) Geometric inequalities and some other properties of the triangle, (7) Parallel lines in a plane, (8) Loci, (9) Space geometry, (10) Proportion and similarity, (11) Areas of polygonal regions, (12) Another modern geometry (Analytical plane geometry) (13) Circles.

We shall include a chapter on 'Other New geometries' wherein we shall very briefly touch upon non-Euclidean geometries so as to give the reader a wider perspective about this very important field of Mathematics—geometry and to inspire him to continue to study it further.

In the various Chapters we will be developing following theorems.

Chapter III. *Incidence postulates and theorems.*

In this chapter we shall develop seven theorems about relationship between points, lines, planes and space.

Chapter IV. *Distance and its measurement.*

In this we will develop the concept of distance between two points and its measurement. We will also develop the 'concept of betweenness'. We will also develop the concept of 'co-ordinate system' on a line. We will also define a segment, a ray and a line. Side by side we will prove some theorems involving betweenness and other order relations. We will also prove two very important theorems, *viz.*, 'The point plotting theorem' and 'The segment construction theorem' which we will use very often in proving other theorems.

Chapter V. *Planes, Space and Separation.*

In this we will talk about 'separation' of a plane into two half planes by a line and separation of space into two half spaces by a plane.

Chapter VI. *Angles and Angle Measurement.*

In this chapter we will define an angle, interior of an angle, right angle, acute angle, complementary angles etc. We will postulate about the measurement and construction of an angle. Then we will prove the following theorems :—

1. (VI/1) Two angles of a linear pair are supplementary and converse.

2. (VI/2) Vertical angles are congruent and its converse.

Chapter VII. *Triangles and Their Congruence.*

We will define a triangle and explain the relationship of congruence between two geometrical figures. We will also define the relationship of congruence between two triangles. We will postulate the S.A.S. congruence and prove theorems some of which are as follows :—

(1) (VII/1) A.S.A. Theorem.

(2) (VII/2) If two sides of a triangle are congruent then the angles opposite these sides are also congruent and its converse.

(3) (VII/3) The perpendicular bisector of the base of an isosceles triangle passes through the vertex opposite that side and converse.

(4) (VII/4) S.S.S. Theorem.

(5) (VII/5) Unique existence of perpendicular from a point P not on the line to the given line.

(6) (VII/6) Every angle has exactly one bisector.

Chapter VIII. *Geometric inequalities and some other properties of triangle.*

We will state Pasch's Axiom and define an exterior angle of a triangle. We will prove the following theorems :—

(1) (VIII/1) The weak form of exterior angle theorem.

(2) (VIII/2) If two sides of a triangle are unequal (not congruent) then the measures of angles opposite these sides are also unequal in the order in which the sides are unequal and converse.

(3) (VIII/3) The shortest segment joining a point to a line is the perpendicular segment.

(4) (VIII/4) If one side of a triangle is atleast as long as either of the other two sides then the foot of the perpendicular from the opposite vertex to the line determined by that side will be an interior point of that side.

(5) (VIII/5) The sum of lengths of any two sides of a triangle is greater than the third side.

(6) (VIII/6) S.A.A. congruency theorem.

(7) (VIII/7) Hypotenuse—leg congruency theorem.

Chapter IX. *Parallel Lines in a Plane.*

We will define parallel lines and prove that two lines are parallel when (*i*) Alternate interior angles are equal (congruent) (*ii*) A pair of corresponding angles are equal (congruent) (*iii*) A pair of interior angles on the same side of the transversal are supplementary.

We will also postulate the existence of parallel lines and prove the converse to the theorems mentioned above. We will also prove the following theorems :—

(1) The sum of the measures of angles of a triangle is 180°.

(2) The sum of the measures of angles of a convex polygon of n sides is $(n-2)\ 180^\circ$.

(3) Opposite sides and angles of a parallelogram are congruent and its diagonals bisect each other and converse.

(4) The diagonals of a rhombus are perpendicular to each other.

(5) If three or more parallel lines intercept congruent segments on one transversal then they intercept congruent segments on any other transversal.

Chapter X. *Loci.*

We will prove the two important theorems on Loci besides proving 'concurrency theorems' on right bisectors of sides, bisectors of angles, the medians and the altitudes of a triangle.

Chapter XI. *Space Geometry.*

In this chapter we will prove another set of 'Incidence theorems' involving the relationship of perpendicularity and parallelism between lines and planes. Some of these are :—

(1) If a line is perpendicular to each of two intersecting lines at their point of intersection then it is perpendicular to the plane that contains these lines and converse.

(2) Through a given point on a given line, there passes one and only one plane perpendicular to the line.

(3) Through a given point in a given plane (and also not in the given plane), there exists one and only one line perpendicular to the plane.

(4) Two planes which are perpendicular to the same line are parallel.

(5) Two lines which are perpendicular to the same plane are parallel.

(6) If a plane intersects two parallel planes, then it intersects them in two parallel lines.

(7) If a line is perpendicular to one of two parallel planes, it is perpendicular to other and converse.

(8) Two planes which are parallel to a third plane are parallel to each other.

Chapter XII. *Proportion and Similarity*.

In this chapter, we will prove some theorems on ratio and proportion. We will also define the relationship of similarity between two triangles and prove (*i*) A.A.A. similarity theorem, (*ii*) A. A. similarity theorem, (*iii*) S.S.S. similarity theorem, and (*iv*) S.A.S. similarity theorem.

We will also prove some other theorems on similarity and then we will prove the Pythagoras theorem and briefly describe 'Pythagorean triplets'.

Chapter XIII. *Area of Polygonal Regions*.

In this chapter we will state 'area postulate' and establish formulae for finding the area of a region bounded by (*i*) a parallelogram, (*ii*) a triangle etc.

Chapter XIV. *Analytical Plane Geometry*.

In this we will build up the system of geometry that is known as 'Analytical plane geometry'. We will state a postulate giving the relationship between the points in a plane and ordered pairs of real numbers. We will establish various equations of a line and conditions of perpendicularity and parallelism between two lines.

Chapter XV. *Circles.*

In this we will define a circle. We will prove various theorems on (*i*) circles including those on intersection or otherwise of a line, and a circle ; (*ii*) tangent circles ; (*iii*) congruence of arcs and chords ; (*iv*) angles inscribed in the same arc ; (*v*) quadrilaterals inscribed in a circle ; (*vi*) addition of arcs ; and (*vii*) the triangle theorem etc. We will also establish formulae for finding circumference and area of a circle.

We will establish equation of a circle in various forms with the methods of analytical plane geometry and the equation and length of a tangent to a circle.

Chapter XVI. In this we will very briefly touch upon non-Euclidean geometries and finite geometries.

CHAPTER III

INCIDENCE POSTULATES AND THEOREMS

A Line. As stated earlier, we will take a 'point', a 'line' or a 'plane' as something undefined. Our 'line' is going to be 'straight' and will extend infinitely in both the directions as in Fig. 14. The arrow-heads on both sides indicate that the line 'does not stop' but extends infinitely in both the directions. We mark any two points on the line as P, Q and denote the line by $\overleftrightarrow{PQ}$. Of course a line will be a collection or set of points.

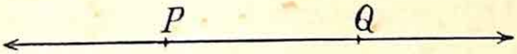


Fig. 14.

A Plane. A plane will be a set or collection of points. We would not try to define it. Instead we will take it as undefined. A geometrical representation of a plane will be the Fig. 15.

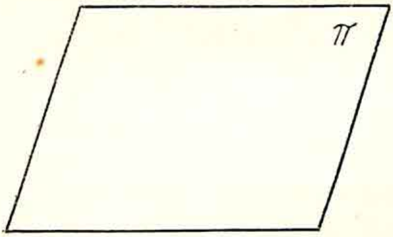


Fig. 15.

We denote it by plane π. Of course our plane is going to extend infinitely in every direction.

Similarly, we will take 'space' as set of all points. In stating Postulates of Incidence, we will make use of the undefined relation 'belong to' along with its variations like 'contains', 'lies on' etc. We shall also make use of the relation of intersection which we define as follows :

Definition III (1). *Intersection of sets.* The intersection of the sets A and B is the set of all elements which are simultaneously members of A and B.

We further state following definitions and postulates.

Definition III (2). *Collinear points.* The points of a set are said to be collinear if there is a line that contains all the points.

Definition III (3). *Non-collinear points.* Three or more points are said to be non-collinear if all of them do not lie on the same line.

Definition III (4). *Coplanar points.* A set of points is said to be coplanar if there is a plane which contains all the points of the given set.

Incidence Postulates

Postulate 1. Every line is a set of points and contains atleast two distinct points.

Postulate 2. If A and B are any two points, then there is one and only one line containing both A and B.

Postulate 3. A plane is a set of points and contains atleast three non-collinear points.

Postulate 4. If A, B, C are any three non-collinear points, then there is one and only one plane that contains all of them.

Postulate 5. If two points of a line lie in a plane, then every point of that line lies in that plane.

Postulate 6. If two planes intersect (have a point in common) then their intersection is a line.

Postulate 7. Space contains atleast four points which are non-collinear and non-coplanar.

The implications of these postulates are :—

(1) The first postulate excludes the possibility of a line consisting of only one point.

(2) The second postulate implies that we cannot have two distinct lines containing the same two given points.

(3) The fifth postulate excludes the possibility of a line entering a plane from one of its sides, going to the other side and then, re-entering the plane from the second side after 'bending' and coming out again to the side from which it entered earlier. In fact this postulate implies that if our planes are 'straight', then our lines will also be 'straight'. If in our physical world the planes are 'straight', then the lines will be straight too !

(4) The sixth postulate excludes the possibility of two planes intersecting in one point only.

(5) The first six postulates tell us only that if points, lines, and planes do exist then what those are going to be. These however, do not tell whether the points, lines or planes exist at all. By stating our seventh postulate, we make sure about the existence of atleast four points. When there exist more than one point, there exists atleast one line, if there exist three non-collinear points, there exists atleast one plane.

Of course the implications of the words 'non-collinear' and 'non-coplanar' may be clearly understood. If we do not use these words then our four points can lie on a line and then there will be no planes or 'space'. If four points are non-collinear but coplanar then these will determine only one plane and hence no space.

It may, however, be stated that the 'line', the 'plane' and the 'space' that we have postulated will each contain an infinite number of points and in the next chapter we will assert about it, but for the purpose that we have in view at the moment, the wording of these postulates is quite adequate. With the help of these postulates, we prove below certain theorems which confirm our common notion about lines, planes, space etc.

Theorem III (**1**). If two different lines intersect then their intersection is atmost one point.

Given : Two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ that intersect.

To prove that : $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect only in one point.

Plane of Proof : We are given that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect and it implies that these have atleast one point in common. We have to prove that these do not intersect in more than one point. We will employ the technique of indirect proof. We will assume that the lines intersect in two points and then we will show that this assumption leads to a contradiction of one of our postulates.

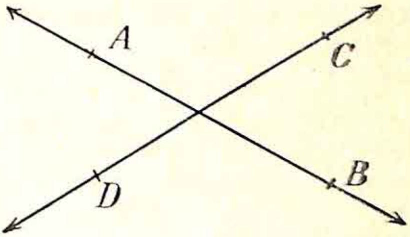


Fig. 16.

Proof : Let $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in two points P and Q.

Then $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have two points P and Q in common. As we are given that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two different lines, therefore, it implies that two different lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have two points in common and this contradicts Postulate 2. Hence the assumption that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have two points in common has to be abandoned.

Hence, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in only one point.

Theorem III (2). If a line that does not lie in a plane but intersects it then the intersection is only a single point.

Given : A line L and a plane π such that the line intersects the plane but does not wholly lie in it.

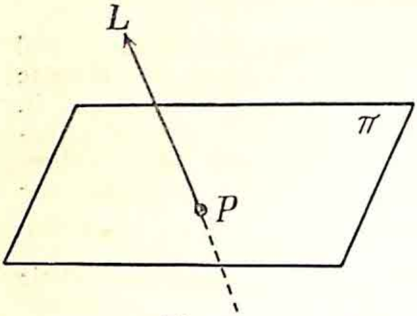


Fig. 17.

To prove that : $\pi \cap L$ is a single point.

Proof: We are given that π and L intersect. So intersection of π and L is atleast one point (P say) or $\pi \cap L = P$. Now we are to prove that $\pi \cap L$ is not more than one point. We will prove it indirectly. We will assume that $\pi \cap L$ are two points P and Q and then we will prove that this leads to a contradiction of some of the postulates or the theorems that we have already proved or what is given (premise).

Assume $\pi \cap L$ are two points P and Q. Then P, Q both belong to π as well as L. (Definition of intersection of two sets, π and L in this case.) Then L wholly lies in π (Postulate 5). This leads to a contradiction as we are given that L does not wholly lie in the plane π. Hence we have to abandon the assumption that $\pi \cap L$ are two points P and Q.

Theorem III (3). Given a line and a point not on the line, then there is one and only one plane that contains both the line and the point.

Given : A line L and a point P such that P does not lie on L or $P \cap L = \phi$.

To prove that : There is one and only one plane that contains both the line and the point or there is one and only one plane that contains $L \cup P$.

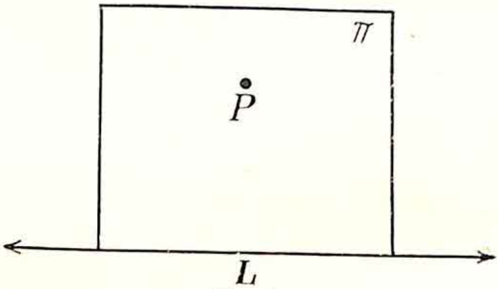


Fig. 18.

Proof :

Statement	*Reason*
(1) L contains at least two points Q and R	(1) Postulate 1
(2) P does not lie on L	(2) Given or premise
(3) P, Q, R are non-collinear	(3) Statements 1 and 2
(4) There is one and only one plane π (say) that contains P, Q, R	(4) Postulate 4
(5) π contains $L \cup P$	(5) Statements 1, 2, 4.

Theorem III (**4**). If two planes intersect, their intersection is exactly one line.

Given : Two planes π and E that intersect (π and E have atleast one point in common).

To prove that : $\pi \cap E$ is exactly one line or $\pi \cap E = \overleftrightarrow{PQ}$ (say).

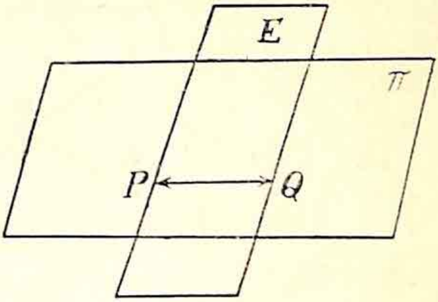


Fig. 19.

Proof :

Statement	*Reason*
1. π and E have a point in common	(1) Premise
2. $\pi \cap E = \overleftrightarrow{PQ}$	(2) Postulate 6.

Now we want to prove that $\pi \cap E$ is only one line. We will prove it indirectly. We will assume that $\pi \cap E$ are two distinct lines and then we will prove that this leads to a contradiction of premise or some postulate.

Assume π and E intersect in two lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$. This implies that $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$ both belong to π and E.

Now there are two possibilities.

(1) $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$ intersect in a point D. $\overleftrightarrow{PQ}$ has a point P ($P \neq D$) and $\overleftrightarrow{AB}$ have a point A ($A \neq D$) and these points P, D, A are non-collinear. Hence P, D, A determine a unique plane (Postulate 4). This contradicts the premise (or what is given to us) that π and E are different planes.

(2) $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$ do not intersect. $\overleftrightarrow{AB}$ has a point C such that $C \notin \overleftrightarrow{PQ}$ (C does not belong to $\overleftrightarrow{PQ}$). $\overleftrightarrow{PQ}$ and C determine a unique plane [Theorem III (3)]. This contradicts the premise that π and E are distinct planes. Hence the assumption that $\pi \cap E$ are two distinct lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$ need to be abandoned. Hence $\pi \cap E$ is only a line.

Theorem III (5). Two intersecting lines determine exactly one plane.

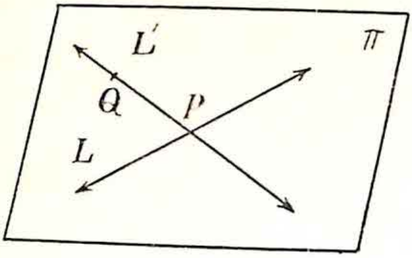


Fig. 20.

Given : Two lines L and L' such that $L \cap L'$ is a point P.

To prove that : $L \cup L'$ determines exactly one plane.

Proof :

Statement	*Reason*
(1) $L \cap L' = P$	(1) Given
(2) $Q \in L'$ such that $Q \neq P$	(2) Postulate 1
(3) $Q \notin L$	(3) L and L' have only one point P in common
(4) Q and L determine a plane π or $Q \cup L = \pi$	(4) Theorem III(3)
(5) π contains P and Q	(5) Statement 4
(6) π contains L'	(6) Postulate 5
(7) π contains $L \cup L'$	(7) Statements 4 and 6.

Now we want to prove that $L \cup L'$ determines only one plane. We will follow the technique of indirect proof.

Assume $L \cup L'$ determines two planes π and E.

It implies that π and E have two lines L and L' in common and this contradicts Theorem 4. Hence the assumption that $L \cup L'$ determines two planes π and E need to be abandoned. Hence $L \cup L'$ determines exactly one plane.

Theorem III (6). Space contains atleast six lines.

Proof :

Statement	*Reason*
(1) Space contains atleast four points which are non-collinear and non-coplanar ; let the points be P, Q, R and S	(1) Postulate 7

(2) Every two points determine one line	(2) Postulate 2
(3) P, Q, R, S taken two at a time determine the lines $\overleftrightarrow{PQ}$, $\overleftrightarrow{PR}$, $\overleftrightarrow{PS}$, $\overleftrightarrow{QR}$, $\overleftrightarrow{QS}$ and $\overleftrightarrow{RS}$	(3) Statements 1 and 2
(4) Space contains atleast six lines	(4) Statement 3.

Theorem III (7). Space contains atleast four planes.

Proof :

Statement	*Reason*
(1) Space contains atleast four points, P, Q, R and S which are non-collinear and non-coplanar	(1) Postulate 7
(2) Every three non-collinear points determine one and only one plane	(2) Postulate 4
(3) P, Q, R, S taken three at a time determine planes $\overleftrightarrow{PQR}$, $\overleftrightarrow{PRS}$, $\overleftrightarrow{PQS}$, $\overleftrightarrow{QRS}$	(3) Statements 1 and 2
(4) Space contains atleast four planes	(4) Statement 3.

Problem Set

1. Verify that the system consisting of five points P_1, P_2, P_3, P_4, P_5 ; ten lines defined as following sets of points :—

$l_1=\{P_1, P_2\}$

$l_2=\{P_1, P_3\}$

$l_3=\{P_1, P_4\}$

$l_4=\{P_1, P_5\}$

$l_5=\{P_2, P_3\}$

$l_6=\{P_2, P_4\}$

$l_7=\{P_2, P_5\}$

$l_8=\{P_3, P_4\}$

$l_9=\{P_3, P_5\}$

$l_{10}=\{P_4, P_5\}$

and ten planes defined as following sets of points :—

$\pi_1=\{P_1, P_2, P_3\}$

$\pi_2=\{P_1, P_2, P_4\}$

$\pi_3=\{P_1, P_2, P_5\}$

$\pi_4=\{P_1, P_3, P_4\}$

$\pi_5=\{P_1, P_3, P_5\}$

$\pi_6=\{P_1, P_4, P_5\}$

$\pi_7=\{P_2, P_3, P_4\}$

$\pi_8=\{P_2, P_3, P_5\}$

$\pi_9=\{P_2, P_4, P_5\}$

$\pi_{10}=\{P_3, P_4, P_5\}$

satisfy the postulates 1-7. Also write:

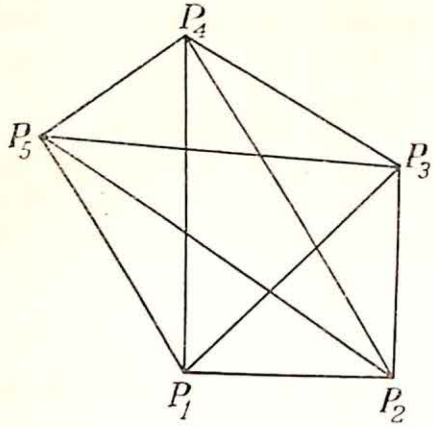


Fig. 21.

(*a*) The lines that lie in the plane π_6.

(*b*) Planes that contain l_9.

(*c*) Planes that pass through P_3.

(*d*) The intersection of π_5 and π_{10}.

(*e*) Construct a physical model of the system with the help of sticks and thread.

II. Assuming that space contains n points, P_1, P_2, $P_3 \ldots P_n$ all different and such that no three of them are collinear and no four of them are coplanar, then how many :

(*a*) Lines will contain any three of them ?

(*b*) Planes will contain any three of them ?

(*c*) Lines the space will contain in all ?

(*d*) Planes the spaces will contain in all ?

III. Assuming we have another Postulate (Postulate 8—Parallel Postulate) which we state as :

"Through a given point not on a given line, there passes one and only one line which is coplanar with the given line but does not intersect it."

With the help of Postulates 1—8, prove the following theorems :

If there exists one line having exactly n points, then :

(*a*) Every line contains exactly n points.

(*b*) Every plane contains exactly $n \times n$ points.

(*c*) Space contains exactly $n \times n \times n$ points.

(*d*) Each plane contains exactly $n(n+1)$ lines.

(*e*) Space contains exactly $n^2(n^2+n+1)$ lines.

(*f*) Space contains exactly $n(n^2+n+1)$ planes.

IV. Verify that space consisting of six points P_1, P_2, P_3, P_4, P_5, P_6 and fifteen lines :

$l_1=\{P_1, P_2\}$
$l_2=\{P_1, P_3\}$
$l_3=\{P_1, P_4\}$
$l_4=\{P_1, P_5\}$
$l_5=\{P_1, P_6\}$
$l_6=\{P_2, P_3\}$
$l_7=\{P_2, P_4\}$
$l_8=\{P_2, P_5\}$
$l_9=\{P_2, P_6\}$
$l_{10}=\{P_3, P_4\}$
$l_{11}=\{P_3, P_5\}$
$l_{12}=\{P_3, P_6\}$
$l_{13}=\{P_4, P_5\}$
$l_{14}=\{P_4, P_6\}$
$l_{15}=\{P_5, P_6\}$

and planes $\pi_1=\{P_1, P_2, P_3, P_4\}$; $\pi_2=\{P_1, P_2, P_5, P_6\}$ and $\pi_3=\{P_3, P_4, P_5, P_6\}$ satisfy postulates 1—7 but does not satisfy Parallel Postulate. Construct a physical model for the system with the help of sticks.

V. A system consists of four points P_1, P_2, P_3, P_4 and lines $l_1=\{P_1, P_2\}$; $l_2=\{P_1, P_3\}$; $l_3=\{P_1, P_4\}$; $l_4=\{P_2, P_3\}$; $l_5=\{P_2, P_4\}$; $l_6=\{P_3, P_4\}$ and planes $\pi_1=\{P_1, P_2, P_3\}$; $\pi_2=\{P_1, P_2, P_4\}$; $\pi_3=\{P_1, P_3, P_4\}$; $\pi_4=\{P_2, P_3, P_4\}$.

Verify that all incidence postulates hold in the system.

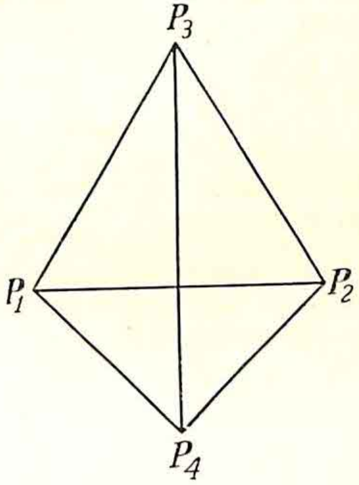


Fig. 22.

Construct a physical model for the same with the help of sticks.

VI. Given points P_1, P_2, $P_3 \ldots P_n$ which are all different such that no three of them are collinear and no four of them are coplanar. Verify that the system satisfies all incidence postulates.

How many lines are there in the system ?

How many planes it has ?

Construct a physical model of the system with the help of sticks by taking $n=8, 10, 11$.

VII. Verify that the system consisting of eight points P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8 ; twenty-eight lines defined as following sets of points:

$l_1 = \{P_1, P_2\}$
$l_2 = \{P_1, P_3\}$
$l_3 = \{P_1, P_4\}$
$l_4 = \{P_1, P_5\}$
$l_5 = \{P_1, P_6\}$
$l_6 = \{P_1, P_7\}$
$l_7 = \{P_1, P_8\}$
$l_8 = \{P_2, P_3\}$
$l_9 = \{P_2, P_4\}$
$l_{10} = \{P_2, P_5\}$
$l_{11} = \{P_2, P_6\}$
$l_{12} = \{P_2, P_7\}$
$l_{13} = \{P_2, P_8\}$
$l_{14} = \{P_3, P_4\}$
$l_{15} = \{P_3, P_5\}$
$l_{16} = \{P_3, P_6\}$
$l_{17} = \{P_3, P_7\}$
$l_{18} = \{P_3, P_8\}$
$l_{19} = \{P_4, P_5\}$
$l_{20} = \{P_4, P_6\}$
$l_{21} = \{P_4, P_7\}$
$l_{22} = \{P_4, P_8\}$
$l_{23} = \{P_5, P_6\}$
$l_{24} = \{P_5, P_7\}$
$l_{25} = \{P_5, P_8\}$
$l_{26} = \{P_6, P_7\}$
$l_{27} = \{P_6, P_8\}$
$l_{28} = \{P_7, P_8\}$

and fourteen planes defined as following sets :

$\pi_1 = \{P_1, P_2, P_3, P_4\}$
$\pi_2 = \{P_1, P_2, P_5, P_6\}$
$\pi_3 = \{P_1, P_2, P_7, P_8\}$
$\pi_4 = \{P_1, P_3, P_5, P_7\}$
$\pi_5 = \{P_1, P_3, P_6, P_8\}$
$\pi_6 = \{P_1, P_4, P_5, P_8\}$
$\pi_7 = \{P_1, P_4, P_6, P_7\}$
$\pi_8 = \{P_2, P_3, P_5, P_8\}$
$\pi_9 = \{P_2, P_3, P_6, P_7\}$
$\pi_{10} = \{P_2, P_4, P_5, P_7\}$
$\pi_{11} = \{P_2, P_4, P_6, P_8\}$
$\pi_{12} = \{P_3, P_4, P_5, P_6\}$
$\pi_{13} = \{P_3, P_4, P_7, P_8\}$
$\pi_{14} = \{P_5, P_6, P_7, P_8\}$

satisfies postulates 1—7. Construct a physical model of the system with the help of sticks.

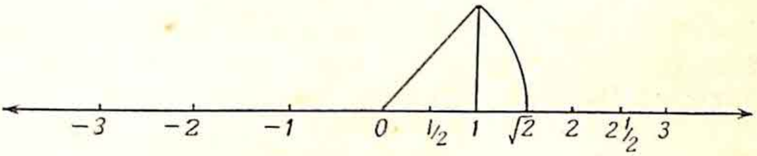
List the planes which contain l_{13}, P_8.

List the lines that lie in $\pi/4$.

CHAPTER IV

DISTANCE AND ITS MEASUREMENT

We have already talked about real number system as an axiomatic system. We can think of real numbers as being arranged as points on a line as in the diagram (given below) which is known as a number line.



Number Line.

Fig. 23.

If our lines in geometry behave like the number line, then we can label the points of any line in geometry with numbers in the same way as we label our number line. To each of the point on our line, there will correspond a real number. In other words, we will be able to set up one-to-one correspondence between the points on our line and the set of real numbers.

One-to-one Correspondence. It may not be out of place to explain to some extent as to what is meant by one-to-one correspondence. Let A be a set of three students. Rami, Kitu and Papu or A={Rami, Kitu, Papu}. They are friends and meet in the club room every evening. In the club room, their are three chairs which are labelled as I, II and III. Suppose, they so decide that Rami always sits in chair labelled I, Kitu in chair labelled II and Papu in chair labelled III. In mathematical terms, Rami and I will form an ordered pair. Similarly, Kitu and II will form an ordered pair and Papu and III will form an ordered pair. In this way we have following three ordered pairs :

S={(Rami, I), (Kitu, II), (Papu, III)}

This set S is a special kind of relation known as one-to-one correspondence. An ordered pair has two components and the relation is a one-to-one correspondence if the first components of each of the ordered pairs are different and also the second component of each of the ordered pairs are different and no component is repeated. In other words, no ordered pair has the same first component or the same second component.

Definition IV (1). A relation is a one-to-one correspondence if and only if (iff) no two ordered pairs of the relation have the same first component or the same second component.

Suppose, the club room has four chairs instead of three and the three friends decide to occupy the chairs in following manner :

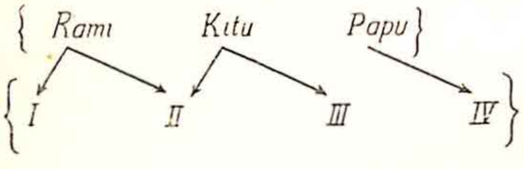


Fig. 24.

It means that Rami will occupy either the chair labelled I or the chair labelled II. Kitu will either occupy the chair labelled II (if unoccupied) or the chair labelled III.

Papu will always occupy chair labelled IV. Now this relation is not a one-to-one correspondence.

Distance. When we are able to set up one-to-one correspondence between the set of points on a line and the set of real numbers then to each pair of points on the line, there will correspond a unique positive number called the distance between the points.

Suppose, we set up one-to-one correspondence between the points of the line $\overleftrightarrow{PQ}$ and the set of real numbers as given below :

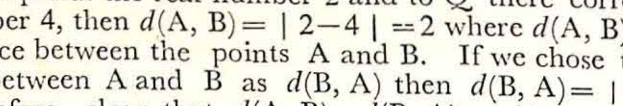


Fig. 25.

If we have two points A, B on $\overleftrightarrow{PQ}$ such that to the point P, there corresponds the real number 2 and to Q there corresponds the real number 4, then $d(A, B) = |2-4| = 2$ where $d(A, B)$ stands for the distance between the points A and B. If we chose to write the distance between A and B as $d(B, A)$ then $d(B, A) = |4-2| = 2$. It is therefore, clear that $d(A, B) = d(B, A)$. Also we state that $d(A, B)$ will be greater than zero wherever A and B are different (or whenever $A \neq B$). In symbols we write it as $d(A, B) > 0$ if $A \neq B$. If $A = B$ or in other words, if A, B are different names for the same points then $d(A, B) = 0$. In symbols we write it as $d(A, B) = 0$, if $A = B$.

We have stated all this about the concept of distance as we do not want to leave anything to common notion. You have been measuring distances and we have not stated here anything different from that. Suppose you want to measure the length of the so-called straight line segment $\overline{AB}$ then you will take your ruler and lay it down in such a manner that the zero of the ruler corresponds to 'A' of the line and if you find that the number 2 corresponds to B then you will say that the length of the line is 2 units. In case you find that your ruler is defective and the 'zero-mark' of the ruler is not visible then you will place the ruler in such a way that '1' of the ruler corresponds to A and then on finding that to point B of the line there corresponds the number 3 on the ruler, you will say that the length of $\overline{AB}=3-1=2$ units. These are the simple ideas that have been stated above in so many words.

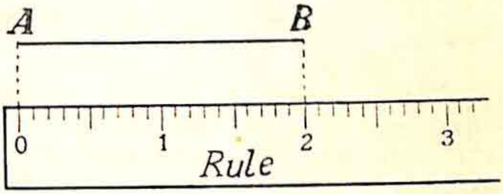


Fig. 26.

When we define the concept of distance in this manner, the common notion of 'straight' of so called 'straight line' automatically comes in. Mathematically speaking our lines are abstract and a mathematician has no concern with the lines or their behaviour in real space. It is the concern of applied scientist. If we consider this question from the view point of a scientist then small distances on our lines are going to be 'straight' but as our lines, according to our definition extend infinitely on both sides, we do not know of what shape our lines are going to be. At the time of Euclid, the planet on which we live, the earth, was considered to be flat, which is not true. Also we do not know of what shape our physical space is. But we as mathematicians have nothing to worry about as our 'lines' will be lines of whatever shape it may be provided those are consistent with the postulates of incidence, postulates of betweenness and other postulates related to lines.

Unit of Distance. To set up one-to-one correspondence between the set of points on a line drawn on the plane of a paper and the set of real numbers we need to have a unit of distance. You have been measuring distances in centimetres or inches etc. The choice of a unit of distance is simply a matter of convenience. One unit of measurement works as well as the other. It will be very convenient for us to agree upon one type of unit of distance. Let us agree that throughout our discussion, the centimetre will be our unit of distance. Whenever we will talk about the measure of any distance, it will be implied that the unit of measurement is a centimetre.

It may be stated that a centimetre is one-hundredth of a metre which is the length of a bar made of an alloy of 90% Platinum and 10% Iridium preserved at International Bureau of Weights and Measures, Paris.

To every pair of points (A, B), there will correspond a number which will indicate the measure of distance between A and B in terms of our unit and which we will indicate as $d(A, B)$ or $m(A, B)$ and which will be read as 'the measure of distance between A and B in terms of our unit'. We will have to talk about distance between two points very often ; so let us decide to write it as AB rather than as $d(A, B)$ or $m(A, B)$. Moreover, whenever we are to say that 'the measure of distance between A, B in terms of our unit is equal to the measure of distance between C, D in terms of our unit', then it will not be correct to write it as $\overline{AB}=\overline{CD}$ as this notation means that the segment $\overline{AB}$ is the same as the segment $\overline{CD}$ which is not the same what our statement meant. However, we can write it as $m(A, B)=m(C, D)$ or $m(\overline{AB})=m(\overline{CD})$, $AB=CD$ or $\overline{AB}\cong\overline{CD}$. $\overline{AB}\cong\overline{CD}$ will be read as the segment $\overline{AB}$ is congruent to segment $\overline{CD}$. We will define a line segment and the relationship of congruence between two line segments later on.

We now state some of the postulates which tell about our common notion of distance.

Postulate 8. (*The Distance Postulate*). To each pair of points in space, there corresponds a unique number such that the number assigned to pair of points is :

(1) 0 if the points of the pair are the same.

(2) Positive if the points of the pair are distinct.

(3) 1 if the points of the pair are at a unit distance. This postulate states that :

(*a*) $d(P, Q)=0$ if $P=Q$.

(*b*) $d(P, Q)=x$ where x is any positive number if $P\neq Q$.

(*c*) $d(P, Q)=1$ if P, Q are at a unit distance.

The number which corresponds to a pair of points in accordance with postulate 8 is called the measure of distance between the points.

Postulate 8 merely asserts that we can associate with a pair of points a unique positive number but it does not indicate how to assign distance measures to pairs of points. If we draw a line segment on the plane of a paper, we measure its length or the distance between its end points with the help of a ruler. For the purposes of geometry that we are developing, we need to have a theoretical ruler. Our theoretical ruler will not have finite number of scale marks. Instead it will be of unlimited extent having infinite number of scale marks. For the purpose of devising this theoretical ruler we state the following postulate.

Postulate 9. (*Ruler Postulate*). The points of a line can be placed in a unique one-to-one correspondence with the set of all real numbers in such a way that :

(1) To every point of the line there corresponds exactly one real number.

(2) To every real number, there corresponds exactly one point on the line.

(3) The measure of distance between any pair of points A and B on the line is equal to the absolute value of the difference of numbers that correspond to A and B.

We have learnt in Algebra that there are infinitely many real numbers. Therefore, by stating postulate 9, we have assured that there are infinitely many points on a line and thus rectified the Euclid's omission. To every point on our line there corresponds a real number and to every real number, there corresponds a point on our line.

The ruler postulate is one of the very important cementing links between Arithmetic and Geometry. The traditional course in Geometry does not have anything like a theoretical ruler. This postulate makes available to geometrical line the order and continuity properties of real number line. This helps us in finding a simpler treatment to the difficult problems of order and continuity which were tacitly assumed by *Euclid.*

Co-ordinate System

The correspondence that we have described above in which we have talked about the theoretical ruler enables us to describe what is known as a 'co-ordinate system'. In fact our theoretical ruler itself is a co-ordinate system.

In a co-ordinate system, we select an arbitrary point and assign to it the number zero. This point is called the origin. Then we find another point which is at a unit distance from the origin. Then we label points corresponding to integers, +1, +2, +3, +4......

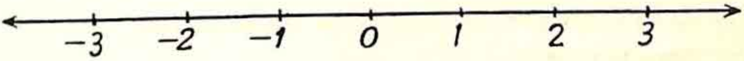


Fig. 27.

and –1, –2, –3...... It is, however, clear that theoretically we can label every point of the line by a unique real number. The number that we assign to a particular point is known as co-ordinate of that point. We can define co-ordinate system as follows :

Definition IV (2). A co-ordinate system on an arbitrary line *l* is a one-to-one correspondence between the set of all points on *l* and the set of all real numbers such that if A and B are points which correspond to 0 and 1 respectively then the measure of distance between any pair of points C and D on *l* relative to the unit of distance that we have chosen, is equal to the absolute value of the difference of the numbers that correspond to C and D.

Order Relations

Now we come to discuss a very simple but very important order relation—betweenness—which helps in ordering the points on a line and which thus helps us in rectifying one more lapse on the part of Euclid. When we think of a point B between the points A and C, it is at onceclear to our mind that the points A, B, and C lie on the same line and the position of the point B in relation to the points A and C would be given by either of the diagrams shown in Fig. 28.

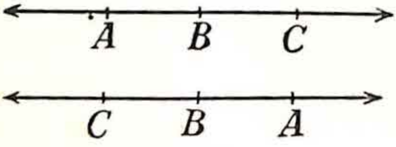


Fig. 28.

Logically speaking both the diagrams tell the same story as on a line there is no way to tell the left from the right. It appears that Euclid regarded the notion of betweenness as very simple and self-evident and although he used it in his proofs yet he failed to define it or mention it in his postulates. In case we do not define it, we have to depend on figure or on spatial intuition. It may be recalled that earlier we have proved that 'every triangle is isosceles'. We could prove it as our construction was wrong. Had we carefully defined betweenness, then we could not have come to that wrong conclusion. We depended on our spatial intuition and arrived at a result which is incorrect. To avoid this dependence on spatial intuition, we will make use of betweenness as a mathematical idea. So we need to define betweenness in exact mathematical terms which we do as follows :

Definition IV (3). (*Betweenness*). A point B is between the points A and C if and only if (Iff) :

(1) A, B, C are distinct and lie on the same line.

(2) $AB+BC=AC$.

This definition simply puts in words our common notion about betweenness. As our common notion goes, the point B will be between the points A and C iff A, B, C are collinear and the sum of measures of the distances between A, B and B, C equals the measure of distance between A and C. Symbolically we write the statement 'the point B is between the points A and C' as A—B—C.

Theorem IV (1). If A—B—C, then C—B—A.

Given : The point B is between the points A and C or A—B—C.

To prove that : The point B is between the points C and A or $C-B-A$.

Proof.

Statement	*Reason*
(1) $AB+BC=AC$	(1) $A-B-C$ (premise)
(2) $BC+AB=AC$	(2) Commutative property of addition as each of BC, AB is a real number
(3) $CB+BA=CA$	(3) $BC=CB$ and $AC=CA$
(4) $C-B-A$	(4) Statement 3 (Def. of Betweenness).

Theorem IV (**2**). If A, B, C are three collinear points and if x, y, z are the co-ordinates of A, B, C respectively then B is between A and C (or $A-B-C$) iff y is between x and z.

Given : A, B, C any three collinear points. The co-ordinates of A, B, C are x, y, z respectively and y is between x and z.

To prove that : $A-B-C$.

Proof. We are given that y is between x and z. Then either

$$x>y>z \text{ or } x<y<z.$$

Let us first take the case when $x>y>z$.

Statement	*Reason*
(1) $x-y>0$ $y-z>0$ $x-z>0$	(1) $x>y>z$
(2) $\lvert x-y \rvert = x-y$ $\lvert y-z \rvert = y-z$ $\lvert x-z \rvert = x-z$	(2) Statement 1 If $x-y>0$ then the absolute value of $(x-y)$ equals $(x-y)$ and so on
(3) $\lvert x-y \rvert = AB$	(3) Ruler Postulate
(4) $\lvert y-z \rvert = BC$	(4) Ruler Postulate
(5) $\lvert x-z \rvert = AC$	(5) Ruler Postulate
(6) $AB+BC= \lvert x-y \rvert + \lvert y-z \rvert$ $=(x-y)+(y-z)=x--z$	(6) Statements 2, 3, 4
(7) $AC= \lvert x--z \rvert = x-z$	(7) Statements 5 and 2
(8) $AB+BC=AC$	(8) Statements 6 and 7
(9) $A-B-C$	(9) Statement 8 (Def. of Betweenness).

We will arrive at the same result if we assume that $x<y<z$.

As the word 'iff' has been used in the statement of the theorem, we have to prove its converse also.

Converse.

Given : A–B–C.

To prove that : y is between x and z.

Proof. We will prove it indirectly by listing down all the possibilities and then eliminating all those which contradict the premise (or what is given).

Now x, y, z are three real numbers assigned to three distinct points on a line so $x \neq y \neq z$. (Distance postulate). So x, y, z must satisfy either one of the following six order relations :

(1) $y>x>z$ (2) $y>z>x$ (3) $z>x>y$

(4) $x>z>y$ (5) $x>y>z$ (6) $z>y>x$.

We are given that A–B–C which means that

$$AB+BC=AC \text{ or } |x-y|+|y-z|=|x-z| \qquad \ldots(i)$$

Now we want to determine which out of six order relations given above is consistent with (i).

Let us first assume that $z>x>y$ then $z-x>0$, $x-y>0$, and $z-y>0$ which means that $x-z<0$, $x-y>0$ and $y-z<0$ which implies that

$$|x-z|=z-x \,;\; |x-y|=x-y \,;$$
$$|y-z|=z-y$$

On substituting these values in (1), we get :

$$(x-y)+(z-y)=z-x$$

or $$x+z-2y=z-x$$

or $$2x=2y$$

or $$x=y$$

or $$A=B$$

which implies that A and B are not distinct points. This contradicts the hypothesis.

Hence the assumption that $z>x>y$ has to be abandoned. Similarly, we can show that $y>x>z$; $x>z>y$; $y>z>x$ are not consistent with the hypothesis. Of course we will find that $x>y>z$ and $z>y>x$ are consistent with (i) and this proves that y is between x and z.

Theorem IV (3). Of the three collinear points, one and only one is between the other two.

Given : Three collinear points A, B, C.

To prove that : If B is between A and C then A will not be between B and C and C will not be between A and B.

Proof. Assume that A$-$B$-$C. Then

$$AB+BC=AC \text{ (Def. of betweenness)} \quad \ldots(i)$$

Now if A is between B and C then BA$+$AC$=$BC should hold and it should be consistent with (*i*) if A$-$B$-$C and B$-$A$-$C are to hold simultaneously.

Now $AB+BC=AC$ (assumption)

or $AC-BC=AB$ ...(*ii*)

But if $BA+AC=BC$ is to be true

then $AC-BC=-BA=-AB$ ...(*iii*)

which is not consistent with the relation (*ii*) and consequently it is not consistent with relation (*i*).

Hence if A$-$B$-$C is true, then B$-$A$-$C will not be true. Similarly, we can show that if A$-$B$-$C holds, then B$-$C$-$A will not hold.

Following are the basic properties of betweenness, some of which we have proved as theorems. Others may be proved by the reader.

(1) If A$-$B$-$C, then C$-$B$-$A.

(2) Of the three collinear points, one and only one is between the other two.

(3) If A$-$B$-$C, then A, B, C are distinct collinear points.

(4) Any four points A, B, C, D can be named on a line in order such that A$-$B$-$C$-$D.

(5) If A, B are any two points, then there is a point C such that A$-$C$-$B and there is a point D such that A$-$B$-$D.

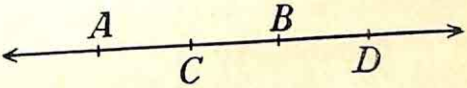


Fig. 29.

The property 5 given above will be used in defining a ray. It is easy to prove it as a theorem. If the co-ordinates of A, B are a and b respectively and if $a<b$, then :

(1) $a+a<b+a$ (Why ?) and $a+b<b+b$ (Why ?).

(2) $2a<a+b<2b$ (statement 1).

(3) $a<\frac{a+b}{2}<b$ (statement 2).

(4) $\frac{a+b}{2}$ is a number, therefore, there corresponds a unique point C to it. *(Ruler Postulate)*

(5) As $a<\frac{a+b}{2}<b$, therefore A—C—B.

(Definition of Betweenness)

Also if $a<b$ then $a<b<b+1$ (Why ?). As $b+1$ is a number, there corresponds a unique point D to $b+1$. Therefore, A—B—D (Why ?).

A Segment and a Ray. After having defined betweenness, we are in a position to define a segment and a ray.

The physical model of a segment PQ will be :

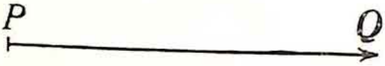


Fig. 30.

The segment PQ consists of the points P, Q and all points between P and Q. We define a segment as under :

Definition IV (4). If P, Q are distinct points, then the set consisting of P, Q and all points which are between P and Q is called the segment PQ. It is denoted as $\overline{PQ}$ or which is same as $\overline{QP}$. In symbols we write this definition of the segment $\overline{PQ}$ as follows :

$\overline{PQ}=\{x : a \leqslant x \leqslant b\}$ where $a<b$ and a, b are respectively the co-ordinates of P and Q.

The distance between the points P and Q is called the length of the segment $\overline{PQ}$.

It may, however, be noted that $\overline{PQ}$ is a set of points while d (P, Q) or m(P, Q) or length of $\overline{PQ}$ or PQ is a number. P, Q are called the end points of the $\overline{PQ}$.

Interior of a Segment. The set of all points of $\overline{PQ}$, excepting the end points P and Q, is called the interior of the segment.

Definition IV (5). (*Congruent segments*). The segments that have the same length are called congruent segments. If the lengths of $\overline{AB}$ and $\overline{CD}$ are equal then $\overline{AB}$ is said to be congruent to $\overline{CD}$. In symbols we write that $\overline{AB}\cong\overline{CD}$. The symbol $\cong$ is made up of two symbols $=$ and $\sim$ and stands for 'equal in size but similar in shape'.

It may be further stated that :

(*a*) Every segment is congruent to itself or the congruency relation between segments is reflexive.

(*b*) If $\overline{AB} \cong \overline{CD}$ then $\overline{CD} \cong \overline{AB}$ or the congruency relation between the segments is symmetric.

(*c*) Segments congruent to same segment are congruent to each other. If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$ or congruency relation between segments is a transitive relation.

The three statements that we have made above can be proved as theorems from the definition of the congruent segments. The proofs are very simple. To an un-initiated, these statements may look as obvious truths but our aim is not to leave anything to intuition. As congruency relation between segments are reflexive, symmetric as well as transitive, we say that for segments, congruence is an equivalence relation.

Ray. A ray is a line extending infinitely in one direction only. Its physical model is

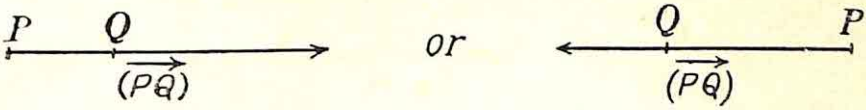


Fig. 31.

$\overrightarrow{PQ}$ means a ray with an end point P and continuing infinitely in the direction of Q. A ray $\overrightarrow{PQ}$ includes all the points of $\overline{PQ}$ and all other points of the line $\overleftrightarrow{PQ}$ which lie on Q-side of P. We define it as follows :

Definition IV (6). A ray $\overrightarrow{PQ}$ consists of set of all points of $\overline{PQ}$ and all points X of $\overleftrightarrow{PQ}$ such that Q is between P and X where X

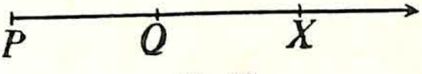


Fig. 32.

is an arbitrary point. In symbols we write it as $\overrightarrow{PQ} = \{x : p \leqslant x\}$ when $p < q$ and p, q are co-ordinates of the points P and Q and x is the co-ordinate of an arbitrary point X of $\overleftrightarrow{PQ}$. If $p > q$ then $\overrightarrow{PQ} = \{x : p \geqslant x\}$.

Another definition of $\overrightarrow{PQ}$ is the union of the segment $\overline{PQ}$ and set of all points X such that P—Q—X.

It may be noted that the definition of betweenness assures that P, Q and the arbitrary point X are collinear. The point P is called the end point of $\overrightarrow{PQ}$. Whenever we denote a ray by symbols, we name the end point of the ray first and the arrow is always extended to the right.

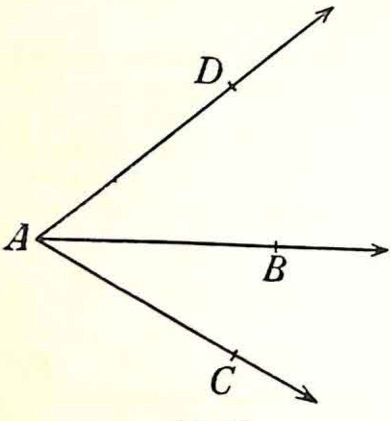


Fig. 33.

Concurrent rays. The rays with the same end point are said to be concurrent. If we have rays $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ with the same end point A, then a physical model representing these would be as in Fig. 33.

Opposite rays. Two concurrent rays which are collinear (lie on the same line) are called the opposite rays and each is said to be opposite the other. Physical model of two opposite rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ will be as in Fig. 34.

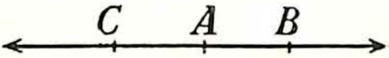


Fig. 34.

It may be noted that $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are not opposite rays. Although these are collinear but their end points are different.

Definition IV (7). (*Opposite rays*). If A is between B and C (C—A—B) then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called opposite rays.

A Line. We have already stated that a line extends infinitely on both the sides. A line AB is denoted by $\overleftrightarrow{AB}$ and its physical model is as in Fig. 35.

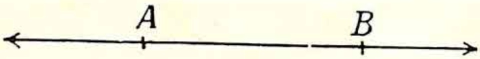


Fig. 35.

If X is an arbitrary point on the line whose co-ordinate is x and if a, b are the co-ordinates of A and B respectively then :

$$\overleftrightarrow{AB} = \{x : x < a \lor a \leqslant x \leqslant b \lor b < x\}$$

which we read as $\overleftrightarrow{AB}$ is a set of points with co-ordinate x such that $x < a$ or $a \leqslant x \leqslant b$ or $b < x$. We can also define $\overleftrightarrow{AB}$ as the union of $\overline{AB}$ set of all points X such that A—B—X and set of all points Y such that Y—A—B.

It may be stated that any two points A and B determine the following six geometric figures :

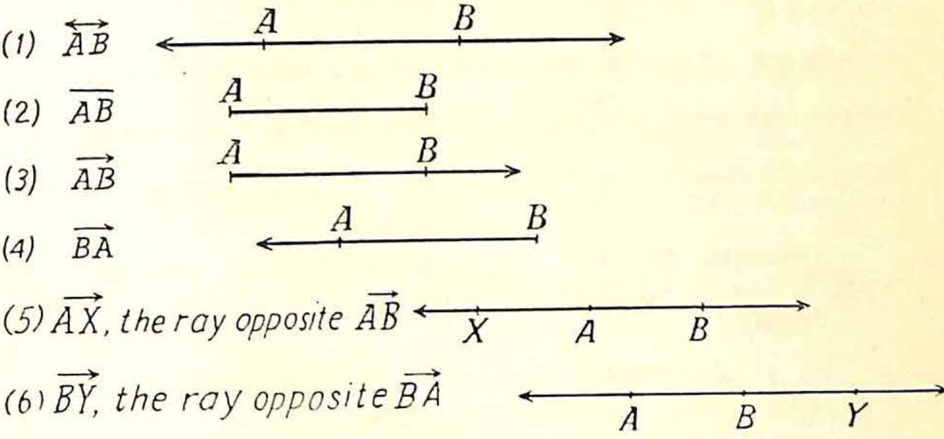

(1) $\overleftrightarrow{AB}$
A
B
(2) $\overline{AB}$
A
B
(3) $\overrightarrow{AB}$
A
B
(4) $\overrightarrow{BA}$
A
B
(5) $\overrightarrow{AX}$, the ray opposite $\overrightarrow{AB}$
X
A
B
(6) $\overrightarrow{BY}$, the ray opposite $\overrightarrow{BA}$
A
B
Y

Fig. 36.

After having defined a segment, a ray and a line we are ready to prove two very important theorems which we will be using very often as we develop our system further.

Theorem IV (4). *(The Point Plotting Theorem)*. If $\overrightarrow{AB}$ is a ray and x is a positive number then there is exactly one point P on $\overrightarrow{AB}$ and one point Q on the ray opposite to $\overrightarrow{AB}$ such that $AP = x$ and $AQ = x$.

Proof. According to the ruler postulate and as per the definition of co-ordinate system, we can choose a co-ordinate system on $\overleftrightarrow{AB}$ such that co-ordinate of A is zero ; there is unique point P whose co-ordinate is x and a unique point Q whose co-ordinate is $-x$ (negative of x). Also :

$$AP = |\,0 - x\,| = x \qquad \textit{(Ruler Postulate)}$$

$$AQ = |\,0 - (-x)\,| = x \qquad \textit{(Ruler Postulate)}$$

Clearly no other point (excepting P, Q) on $\overleftrightarrow{AB}$ can be at a distance of x from A. Also as these points are on the opposite rays on $\overleftrightarrow{AB}$, one of them will be on $\overrightarrow{AB}$ and the other will be on the ray opposite to $\overrightarrow{AB}$.

Theorem IV(5). (*The Segment Construction Theorem*). Given a segment $\overline{AB}$ and a ray $\overrightarrow{PQ}$, there is exactly one point C on $\overrightarrow{PQ}$ such that $\overline{AB} \cong \overline{PC}$.

Proof. As in the case of point plotting theorem, we choose a co-ordinate system on $\overrightarrow{PQ}$ such that co-ordinate of P is 0. There will be a unique point C on $\overrightarrow{PQ}$ such that the co-ordinate of C is the number AB. (AB is the length of $\overline{AB}$ and AB is a number.)

Theroem IV (6). (*The Segment Addition Theorem*). If $A-B-C$ and $P-Q-R$ and if $\overline{AB} \cong \overline{PQ}$ and $\overline{BC} \cong \overline{QR}$ then $\overline{AC} \cong \overline{PR}$.

Proof.

Statement	*Reason*
(1) $AB+BC=AC$	(1) $A-B-C$
(2) $PQ+QR=PR$	(2) $P-Q-R$
(3) $\overline{AB} \cong \overline{PQ}$ and $\overline{BC} \cong \overline{QR}$	(3) Given
(4) $AB+BC=PQ+QR$	(4) Statement 3 (Why E_1 ?) (AB, BC, PQ, QR are numbers)
(5) $\overline{AC} \cong \overline{PR}$	(5) Statements 1 and 2 (As $AC=PR$).

The following is the converse to the segment addition theorem.

Theorem IV (7). (*The Segment Subtraction Theorem*).

If $A-B-C$ and $P-Q-R$ and if $\overline{AB} \cong \overline{PQ}$ and $\overline{AC} \cong \overline{PR}$ then $\overline{BC} \cong \overline{QR}$.

This can be proved on similar lines as Theorem IV(3).

Definition IV(8). (*Mid-point*). A point M is called the mid-point of $\overline{AB}$ if $A-M-B$ and if $AM=MB$ or $\overline{AM} \cong \overline{MB}$.

Theorem IV (8). Every segment has exactly one mid-point.

Given : A segment $\overline{AB}$.

To prove that : There is unique point M on $\overline{AB}$ such that $A-M-B$ and $AM=MB$.

Proof.

Statement	*Reason*
(1) There is a unique point M on $\overrightarrow{AB}$ such that $AM=\frac{1}{3}AB$	(1) Point plotting theorem
(2) $A-M-B$	(2) $AM<AB$ as $AM=\frac{1}{2}AB$

(3) AM+MB=AB	(3) Statement 2
(4) MB=$\frac{1}{2}$ AB	(4) Statements 1 and 3
(5) AM=MB	(5) Statements 1 and 4
(6) M is mid-point of $\overline{AB}$	(6) A–M–B and AM=MB.

Definition IV(9). The mid-point of a segment is said to bisect the segment. In general any set of points (any figure) whose intersection with a segment consists only of mid-point of the segment is said to bisect the segment.

Problems Set

1. Simplify: (*a*) $|7-9|$ (*b*) $|-7-9|$ (*c*) $|7-(-9)|$ (*d*) $|-7-(-9)|$ (*e*) $|3-10|$ (*f*) $|-3-10|$ (*g*) $|-3-(-10)|$

2. You have some pairs of points on a line. You arbitrarily choose any point on the line and assign to it the number zero and then set up a co-ordinate system. Find the distance between the pairs of points to whom you have assigned the following co-ordinates:

(*a*) 3 and 10 (*b*) 3 and −10 (*c*) −3 and −10 (*d*) −3 and 10 (*e*) 0 and −7 (*f*) 0 and+7 (*g*) 7·2 and −7·2 (*h*) $x-y$ and $x+y$ (*i*) 8 and 0 (*j*) —8 and 0 (*k*) --$3\frac{1}{2}$ and $-3\frac{1}{2}$ (*l*) $-\frac{3}{2}x$ and $\frac{3}{2}x$.

3. Assume that you have built up a coordinate system and assigned the following coordinates to pairs of points on a line:

(*a*) --7 and −10 (*b*) 7 and −10 (*e*) 7 and +10 (*f*) −7 and −10.

Now suppose that you decide to assign the number zero to the first point in each pair. Find the co-ordinate that you will assign to the second point in each case.

4. Given three collinear points A, B, C and AB=15 and BC=25, then:

(*a*) If A–B–C, find AC.

(*b*) If B–A–C, find AC.

(*c*) Is it possible to have B–C–A? If not, why not?

5. Given three collinear points A, B, C where AB=5 and BC=10. In how many ways can you arrange these on the line? Illustrate with the help of a physical model.

6. If A–B–C and A–C–D, then prove that A–B–D.

7. If A–B–C and B–C–D, then A–B–D and A–C–D.

8. If A–B–C, then $\overline{AB}$ and $\overline{BC}$ have only one point B in common.

9. Find the co-ordinates of mid-point of $\overline{AB}$ if the co-ordinates that you assign to A and B are as follows:

(*a*) 2 and 4 (*b*) -2 and 4 (*c*) -2 and -4 (*d*) 2 and -4. Draw physical model in each case.

10. If $A-B-C$, then $AC>AB$ and $AC>BC$.

11. Find the intersection of (*i*) $\overline{AB}$ and $\overrightarrow{AB}$ (*ii*) $\overline{AB}$ and $\overrightarrow{BA}$.

12. Prove the segment subtraction theorem.

13. If $A-B-C$ then A, B, C are distinct collinear points.

14. Prove that any four points A, B, C, D can be named on a line in order such that $A-B-C-D$.

15. If A, B are any two points then $\overline{AB \cong BA}$.

16. If $\overline{AB}=\overline{CD}$ then $A=C$ and $B=D$.

17. If R is a point on $\overrightarrow{PQ}$ such that $R \neq P$ then $\overrightarrow{PQ}=\overrightarrow{PR}$.

18. Four collinear points A, B, C, D are such that $AC>AB$ and $BD<BC$. Draw as many physical models of this as possible.

19. Which of the following is (*i*) a point (*ii*) a ray (*iii*) a line if x is the co-ordinate of an arbitrary point on a line :

(*i*) $x=-7$ (*ii*) $x<-7$ (*iii*) $x \leqslant -7$ (*iv*) $|x| \leqslant -13$ (*v*) $|x|<7$ (*vi*) $|x|>7$ (*viii*) $|x| \geqslant 0$?

[The set of points given by $x<-7$ is known as a half line. A half line is a ray without its end point. The set of all the points constituting the ray $\overrightarrow{AB}$ but without its end point A is known as half line AB which is symbolically represented as $\overset{\circ\!\rightarrow}{AB}$. We can define it as $\overset{\circ\!\rightarrow}{AB}=\{P : P=B \text{ or } A-P-B \text{ or } A-B-P\}$].

CHAPTER V

PLANES, SPACE AND SEPARATION

In the previous chapter, we talked about lines, planes, space etc. We carefully developed the concept of betweenness so that we may not have to assume certain properties on the basis of figure. But our preparation is still incomplete. We also need to develop the concept of separation. The concept of separation includes (*i*) point separation (*ii*) plane separation (*iii*) space separation.

We have intuitive notion that any point on a line divides or separates the line into two parts—two opposite rays with that point as the common end point. The point A on a line *l*, separates it into two opposite rays $\overrightarrow{AD}$ and $\overrightarrow{AB}$ as in this figure. In other words, the point A separates the rest of the points of the set of the points that constitute *l* into two sub-sets which we call as sides of A. If the point A is not to be included in any of the sub-sets, then these two sub-sets will be the two half lines into which A separates *l*. In this figure, the segment $\overline{CD}$ whose end points are on the same side of A is said to lie entirely on that side of A. Further, we can say that $\overline{CD}$ lies on C-side of A or D-side of A. We will be often making use of this notation. A segment like $\overline{BD}$ whose end points lie on the opposite sides of A contains A in the interior. Summarizing we state the following :

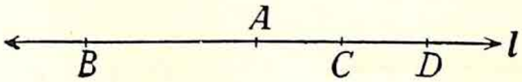


Fig. 37.

A point separates a line into two parts in such a way that :

(*a*) Any segment such as $\overline{CD}$ whose end points lie on the same side of A lies entirely on that side of A.

(*b*) Any segment such as $\overline{BC}$ whose end points lie on the opposite sides of A contains A in its interior.

The natural corollary that arises out of this discussion is : 'If a point separates a line into two parts then will a line separate a plane into two parts ?'. The axioms and postulates that Euclid has stated, do not help us in this regard. Also the postulates that we have stated so far do not help. So we have to state a postulate for this purpose. Before we do so, we have to develop a very important concept—the concept of convex sets.

Convexity of Sets. Given a set S. If we arbitrarily take any two points A, B in S and if the entire segment lies $\overline{AB}$ in the given set S, then S is said to be convex. Following are examples of convex sets :—

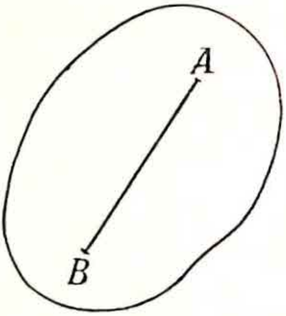
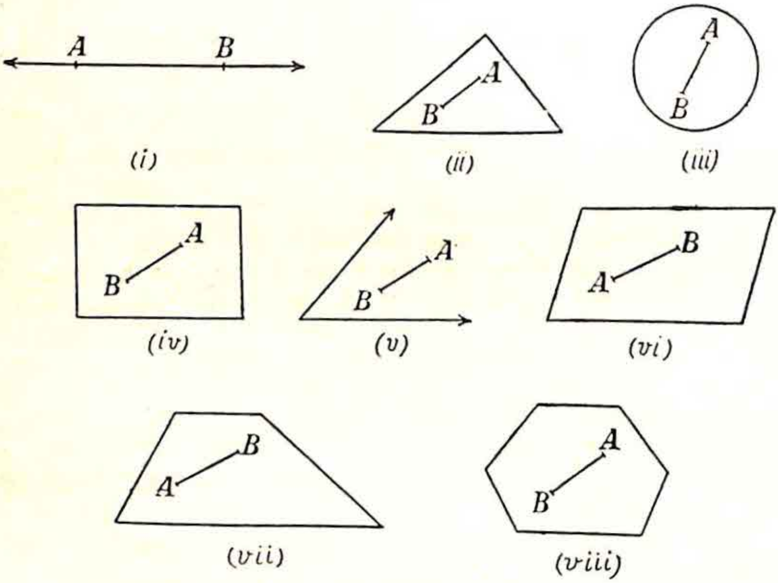


Fig. 38.



(i) (ii) (iii) (iv) (v) (vi) (vii) (viii)

Fig. 39.

Fig. 40 on next page consists of physical models of sets which are not convex.

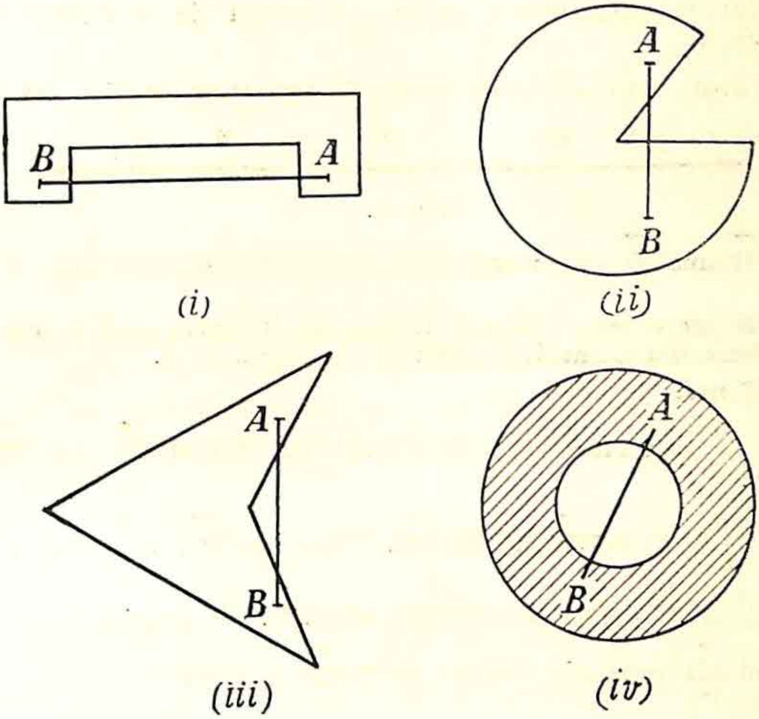
To show that set S is not convex we have to arbitrarily take any two elements A, B of S and to show that $\overline{AB}$ does not wholly lie in S.

A set consisting of only one point is convex as you can take only one point (say A) in the set and $\overline{AA}$ belongs to set. A line is a convex set, a plane is a convex set, space is a convex set. We define a convex set as follows :

Definition V (1). A set of points is convex iff any segment whose end-points both belong to the set entirely lies in the set.

Following theorem explains the most fundamental property of convex sets.

Theorem V (1). The intersection of two convex sets is also a convex set.



(i) (ii) (iii) (iv)

Fig. 40.

Given : S_1 and S_2 any two convex sets.

To prove that : $S_1 \cap S_2$ is a convex set.

Proof. Let A, B be any two points in $S_1 \cap S_2$.

Statement	*Reason*
(1) A, B both belong to S_1	(1) As A, B both belong to $S_1 \cap S_2$ (definition of intersection)
(2) A, B both belong to S_2	(2) As A, B both belong to $S_1 \cap S_2$ (definition of intersection)
(3) $\overline{AB}$ entirely lies in S_1	(3) S_1 is a convex set
(4) $\overline{AB}$ entirely lies in S_2	(4) S_2 is a convex set
(5) $\overline{AB}$ entirely lies in $A \cap B$	(5) Statements 3 and 4 and definition of intersection
(6) $A \cap B$ is a convex set	(6) Statement 5 and also as A, B belong to $S_1 \cap S_2$.

Now we discuss a theorem regarding the separation of a line by a point.

Theorem V (2). A point P lying on a line *l* divides the rest of points on the line into two sets such that :

(1) Each set is convex.

(2) Any segment of *l* whose end points lie in different sets contains A.

Given : A line *l* and a point $P \in l$ separating the line into two

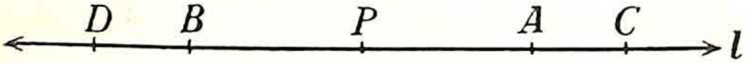


Fig. 41.

sets $\overset{\circ\rightarrow}{PC}$ and $\overset{\circ\rightarrow}{PD}$ (of course P is not to be included into any of the sets).

To prove that : Each of the two sets is convex and a segment $\overline{AB}$ whose end points lie in different sets, contains A.

Proof.

(1) Postulate 2 helps us to assert that each of $\overrightarrow{PA}$ and $\overrightarrow{PB}$ is convex.

(2) Now $A \in \overrightarrow{PC}$ and $B \in \overrightarrow{PD}$. Also $\overline{AB} = \overrightarrow{BC} \cap \overrightarrow{AD}$ (Why ?).

Also $\overrightarrow{BC} = \overline{BP} \cup \overrightarrow{PC}$ and $\overrightarrow{AD} = \overline{AP} \cup \overrightarrow{PD}$. It implies that both $\overrightarrow{BC}$ and $\overrightarrow{AD}$ contain P. Hence $\overrightarrow{BC} \cap \overrightarrow{AD}$ contains P. Hence $\overline{AB}$ contains P.

Now we state a postulate regarding the separation of a plane by a line.

Postulate 10. (*The Plane Separation Postulate*). Given a line and a plane containing it, the points of the plane which do not belong to the line form two sets such that :

(1) Each set is convex.

(2) A segment having one of the end points in one set and the other end point in the second set, intersects the given line.

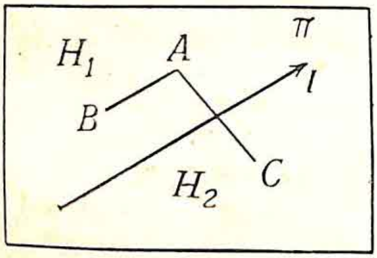


Fig. 42.

The physical model representing this postulate is as in Fig. 42. A line *l*, lying in the plane π separates the set of points in the plane into two sets H_1 and H_2. If we take any segment AB whose end points both lie in H_1 then the segment $\overline{AB}$ wholly lies in H_1 implying that H_1 is convex. But if we take a point C in H_2 then $\overline{AC}$ intersects *l*.

Each of the convex sets H_1 and H_2 is called a half plane. The line l which according to the plane-separation postulate does not belong to either H_1 or H_2 is called the edge of each half plane and is said to separate the plane into two half planes. Two or more points which lie on the same half plane are said to lie on the same side of the given line. The two points that lie in different half planes are said to lie on opposite sides of the given line. There will be infinitely many half planes having the line l for its edge, some of which are shown in Fig. 43.

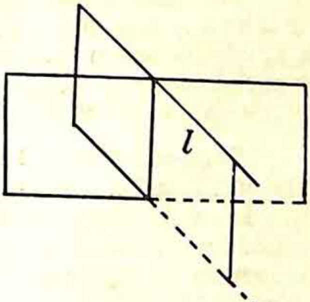


Fig. 43.

A plane separates the space into two half-spaces such that each half-space is convex and any segment joining a point in one half-space to another in the other half-space intersects the plane. We state the following postulate containing these notions.

Postulate 11. (*The Space Separation Postulate*). Given a plane in space. The set of points of space which do not lie in the given plane form two sets such that :

(1) Each set is convex.

(2) Any segment joining a point in one set to a point in the other set, intersects the given plane.

The two sets into which a plane separates the space are known as half-spaces. We have included this postulate to simplify the matters ; otherwise this can be proved as a theorem. Its proof as a theorem is rather complicated. However, it is given below :

Theorem V (3). (*Space Separation Theorem*). Given a plane P in space. The set of points which do not lie in the plane P form two sets S_1 and S_2 such that :

(1) Each of S_1 and S_2 is convex.

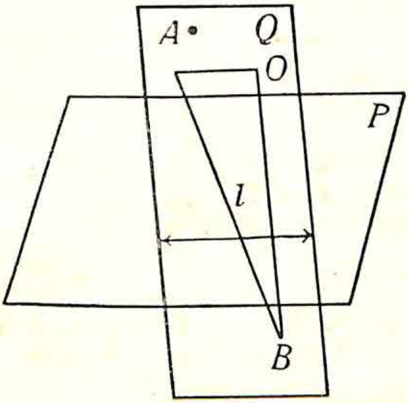


Fig. 44.

(2) If A is a point in S_1 and B is a point in S_2 then the segment $\overline{AB}$ intersects the plane P.

Proof. Let the given plane P divide the set of points of the space which do not belong to P into two sets S_1 and S_2 in the following manner.

(1) Set S_1 consisting of an arbitrary point O not belonging to the given plane and all other points like A such that segment $\overline{OA}$ does not intersect P.

(2) The set S_2 consisting of all points like B not belonging to the given plane P such that the segment $\overline{OB}$ intersects the given plane P.

We want to show that S_1 and S_2 are convex. It means that if we take any two arbitrary points A_1 and A_2 in S_1, then the segment $\overline{A_1A_2}$ wholly lies in S_1. Similarly, if we take points B_1 and B_2 in S_2 then the segment $\overline{B_1B_2}$ wholly lies in S_2. Also the segment $\overline{A_1B_1}$ intersects the plane P.

Suppose, A and B are arbitrary points in S_1 and S_2 respectively. The segment $\overline{OB}$ intersects the plane P according to our definition of S_2. Let C be the point of intersection. Let Q be the plane determined by O, A and B or any plane containing O, A, B if these happen to be collinear (Postulate 3). Now the planes P and Q will intersect in a line (l say) (Postulate 6). In the plane Q, the points O and A are on the same side of the line l as $\overline{OA}$ does not intersect the plane P. (Def. of S_1). But in the plane Q, the points O and B lie on the opposite side of l as $\overline{OB}$ intersects the plane P (Def. of S_2). Therefore, A and B are on opposite side of l and hence the segment $\overline{AB}$ intersects the line l and hence the plane P (as $l \subset P$).

Now we want to show that S_1 is convex. Let A_1, A_2 be any two points in S_1. Now there are following three possibilities :

(1) The segment A_1A_2 does not entirely lie in S_1 and has a point D which belongs to S_2. The physical model of the segment $\overline{A_1A_2}$ then will be as in Fig. 45.

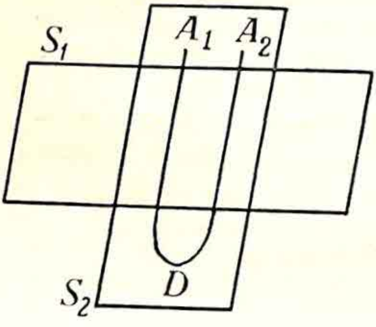


Fig. 45.

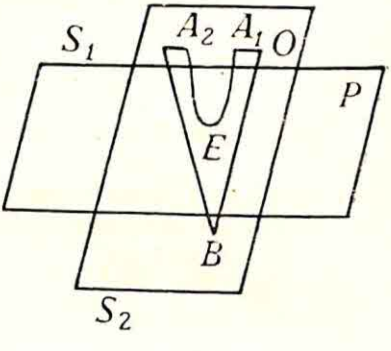


Fig. 46.

(2) The segment A_1A_2 does not entirely lie in S_1 and the segment $\overline{A_1A_2}$ has a point E which belongs to P. The physical model of the segment $\overline{A_1A_2}$ in this case will be as in Fig. 46.

(3) The segment $\overline{A_1A_2}$ entirely lies in S_1.

Now if A_1A_2 contains a point $D \in S_2$ then by the first part of our proof both $\overline{A_1D}$ and $\overline{A_2D}$ will each have a point in common with P. Moreover, as these will lie on opposite rays $\overrightarrow{DA_1}$ and $\overrightarrow{DA_2}$, these points will be distinct. Hence $\overline{A_1A_2}$ will have two points in common with π and hence $\overline{A_1A_2}$ must entirely lie in π (Postulate 5) which is not possible as A_1, A_2 are elements of S_1 and according to our definition of S_1, it does not contain any element of P. Hence the first possibility has to be rejected.

Let us now consider the second possibility. $\overline{A_1A_2}$ contains a point $E \in P$. Then the plane containing O, A_1, A_2 having E in common with P must intersect P in a line (l' say) and in this plane A_1 and A_2 must lie on the opposite side of l' and hence $\overline{OA_1}$ or $\overline{OA_2}$ must intersect l' and hence P (Plane Separation Postulate). This is impossible in view of our definition of S_1. Hence this possibility also stands rejected. Hence $\overline{A_1A_2}$ entirely lies in S_1. Therefore, S_1 is a convex set.

Similarly, we can prove that S_2 is a convex set. Thus the theorem is proved.

Each of S_1 and S_2 is called a half-space. The plane P is called the face of each half-space. Two or more points in the same half-space are said to lie on the same side of the given plane. Two points which lie in different half-spaces are said to lie on opposite sides of the given plane.

Some Incidence Theorems based on Plane Separation Postulate.

Below we discuss some incidence theorems based on plane separation postulate .

Theorem V (4). If A and B are on the opposite sides of a line l and if (*i*) B and C are on opposite sides of l then A and C are on the same side of l. (*ii*) B and C are on the same side of l then A and C are on opposite side of l.

Given : A and E are on opposite sides of l. Also B and C are on opposite sides of l.

To prove that : A and B are on the same side of l.

Proof. Let l divide the plane π in two half planes H_1 and H_2.

Statement	*Reason*
(1) If A is in H_1 then B is in H_2	(1) A and B are on opposite side of l
(2) If B is in H_2 then C is in H_1	(2) B and C are on opposite side of l
(3) A and C are in H_1	(3) Statements 1 and 2
(4) A and C are on same side of l	(4) Statement 3

(*ii*) *Given* : A and B are on opposite side of l. B and C are on same side of l.

To prove that : A and C are on opposite side of *l*.

Proof.

Statement	*Reason*
(1) If A is in H_1 then B is in H_2	(1) Why ?
(2) If B is in H_2 then C is in H_2	(2) Why ?
(3) If A is in H_1 then C is in H_2	(3) Statements 1 and 2.
(4) A and C are on opposite side of *l*	

Theorem V (5). Given a line *l* and a ray whose end point is on *l*. Prove that all the points of the ray except the end point lie on the same side of *l*.

Given : A line *l* and a ray $\overrightarrow{PQ}$ whose end point P is on *l*.

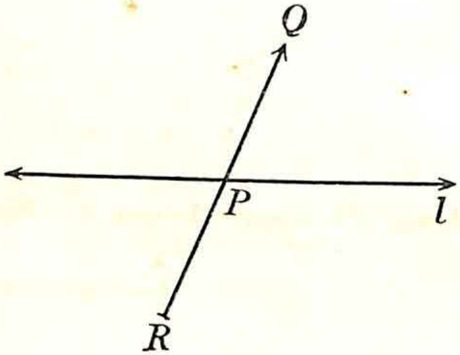


Fig. 47.

To prove that : All points of $\overrightarrow{PQ}$ excepting P lie on the same side of *l*.

Proof. We will prove it indirectly by contradiction. Assume that *l* divides the plane of the line into two half planes H_1 and H_2. Assume that Q lies in H_1. Further assume that $\overrightarrow{PQ}$ contains a point $R \in H_2$. In other words we are assuming that the set of points $\overrightarrow{PQ}$ excepting P does not wholly lie in H_1. As Q and R are on opposite side of *l*, therefore $\overline{QR}$ intersects *l*. The point of intersection will be P as $\overline{QR}$ lies on $\overrightarrow{PQ}$ and P lies on *l*. Also as *l* and $\overline{QR}$ cannot intersect in more than one point (Why ?). Therefore, R—P—Q (Why ?) which implies that R is the end point of $\overrightarrow{PQ}$ which contradicts the definition of $\overrightarrow{PQ}$ as P is the end point of $\overrightarrow{PQ}$. Hence the assumption that $\overrightarrow{PQ}$ contains a point $R \in H_2$ has to be abandoned.

CHAPTER VI

ANGLE AND ANGLE MEASUREMENT

In elementary geometry, an angle has been sometimes defined in terms of amount of rotation or in terms of difference of direction between two lines. These definitions are not very satisfactory as it is not possible to define rotation or direction in exact mathematical terms. Also it is not easy to put in clear words as to how the rotation takes place. Consider the two angles whose physical models are given below :

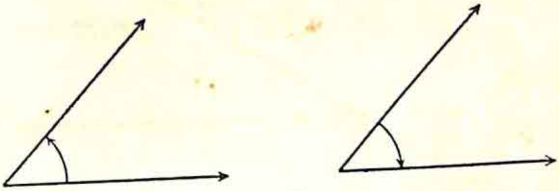


Fig. 48.

The ray that serves as the initial side in the first case is in the position of the ray that serves as terminal side in the second case. Also the rotation in two cases are in different directions. So we will have to consider these angles being different although their measures may be equal. We may have to introduce the notion that measure of an angle is positive when the rotation is counter-clockwise and it is negative when the rotation is clockwise.

Another approach is to define an angle in terms of some set of points. This method of defining an angle solves some of our problems that we encounter while defining an angle in terms of amount of rotation or the change in direction. This mode of defining an angle has been commonly accepted and we will employ this very approach and define an angle in the following manner.

Definition VI (1). (*Angle*). An angle is the union of two rays which have a common end point and which do not lie in the same line.

Each of two rays which form the angle are called the sides of the angle and their common end point is called the vertex of the angle. The angle that is formed by the union of rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is either denoted by $\angle BAC$ or $\angle CAB$. We can denote an angle by its vertex if there is no possibility of any confusion. In that case we can denote $\angle BAC$ or $\angle CAB$ as simple $\angle A$.

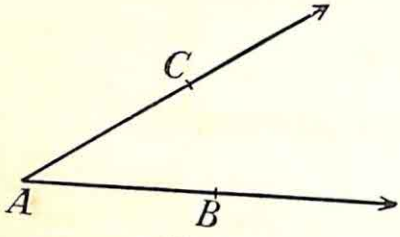


Fig. 49.

It may also be noted that we have used the word 'ray' in our definition of an angle. It implies that the sides of our angles extend infinitely and we show it with the help of arrow heads. Also according to our definition an angle consists of union of two rays and hence it does not include the points that lie within the two sides of the angle. The set of such points is known as the interior of the angle.

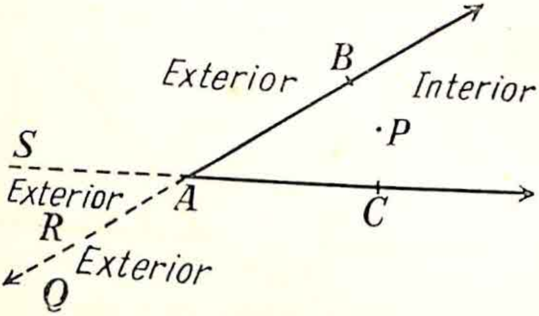


Fig. 50.

In Fig. 50 the set of points like P belongs to interior of the angle A. The points like B, A or C belong to angle A. The points like Q, R, S belong to the exterior of $\angle A$. In fact the set of points belonging to the plane of this paper have been divided into three sets S_1, S_2 and S_3 such that :

(1) The set S_1 consisting of all points like P such that P lies on the same side of $\overrightarrow{AC}$ as B and on the same side of $\overrightarrow{AB}$ as C. In other words the points like P lie on E–side of $\overrightarrow{AC}$ and C–side of $\overrightarrow{AB}$. These points constitute the interior of $\angle A$.

(2) The set S_2 consisting of all points lying on the union of the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. These points constitute the $\angle A$.

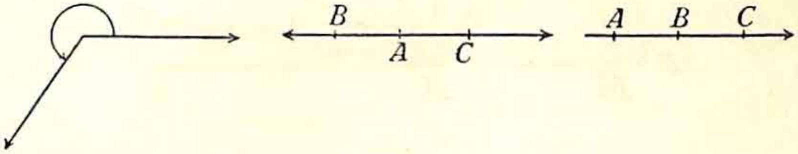
(3) The set S_3 consisting of all those points of the plane which neither belong to the interior of the angle nor to the angle.

We define the interior and the exterior of the angle as follows :

Definition VI (2). (*Interior of an angle*). If an $\angle BAC$ lies in a plane π, then a point P of the plane lies in the interior of the $\angle BAC$ if P lies on the B—side of $\overrightarrow{AC}$ and C—side of $\overrightarrow{AB}$.

Definition VI (3). (*Exterior of an Angle*). The exterior of $\angle BAC$ is the set of all those points of the plane π that neither belongs to the interior of the $\angle BAC$ nor to the $\angle BAC$. ('π' is of course the name of the plane in which the angle lies.)

It may also be noted that figures that are referred to as 'straight angles' or 'zero-angles' or 'reflex-angles' are not angles according to the definition of the angle that we have adopted. So the diagrams in Fig. 51 are not angles according to our definition.



(So called reflex angle) (So called straight angle) (So called zero angle)

Fig. 51.

These diagrams do have some importance in a course on Trigonometry but these do not serve much useful purpose in elementary Geometry. These have been purposely excluded as to include them would necessitate creating exceptions to various fundamental theorems that we are going to take up.

Measurement of an angle. The angles are generally measured in degrees. It may be clearly understood that 'an angle' and 'measure of an angle' are two different things. The angle is a union of two rays while the measurement of an angle is the number of degrees that it contains.

If there are 30° in an $\angle BAC$, then it will be denoted as $m\angle BAC=30$. In writing so, we do not indicate the unit of the measure as we understand that it will be degrees. Whenever we are to express the measure of an angle in a unit different from degrees, we will indicate the same.

We measure an angle with the help of a protractor. Let $\overrightarrow{AB}$ be one side of the angle that we want to measure. Then $\overleftrightarrow{AB}$ divides the plane of the paper into two half planes H_1 and H_2. Assume that the angle that we want to measure lies in H_1. We will measure the angle by placing the edge of the protractor on $\overleftrightarrow{BA}$, the edge of the half plane H_1. In Fig. 52 :

$m\angle CAB=10$
$m\angle DAB=30$
$m\angle EAB=45$
$m\angle FAB=60$
$m\angle GAB=90$
$m\angle HAB=120$, and so on.

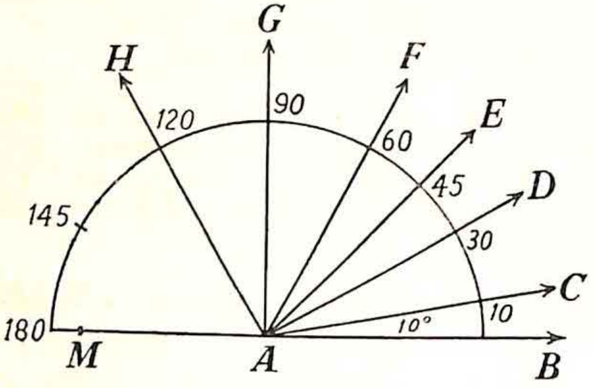


Fig. 52.

Note that there will be no such angles as $\angle MAB$ as $\overrightarrow{AB}$ and $\overrightarrow{AM}$ are collinear.

Now we state some postulates regarding measure of angles which are merely abstract descriptions of the familiar behaviour of the protractor.

Postulate 12. (*The Angle Measurement Postulate*). To every angle A, there corresponds a real number between 0 and 180.

This postulate, of course, clearly lays down that measure of an angle will be greater than zero but less than 180.

Postulate 13. (*Angle Construction Postulate*). If $\overrightarrow{AB}$ is any ray on the edge of half plane H_1, then for every real number r between zero and 180, there is exactly one ray $\overrightarrow{AK}$ with K in the half plane H_1 such that $m\angle KAB=r$.

This postulate states that to every real number between 0 and 180, there corresponds one and only one ray in H_1 while the angle measurement postulate states that to every ray in H_1, there corresponds a real number between 0 and 180. Postulate 12 helps us in measuring an angle. We denote the measure of an $\angle A$ by $m\angle A$. The statements like $\angle A=30$ do not have any meaning while one like $m\angle A=30$ states that the measure of $\angle A$ is thirty degrees. We state another postulate to assert that the sum or difference of measures of two angles is measure of another angle.

Postulate 14. (*The Angle Addition Postulate*). If P is a point in the interior of $\angle BAC$, then :

$$m\angle BAC=m\angle BAP+m\angle PAC$$

Spposing we want to find the measure of $\angle$BAP, we can either place the edge of the protractor on $\overrightarrow{AC}$ or we can find the measure of $\angle$BAC and $\angle$PAC and then with the help of postulate 14, we can find the measure of $\angle$BAP by 'subtraction'.

We could have combined postulates 12 and 14 and then we could have proved postulate 13 as a theorem. This could have been done as follows :

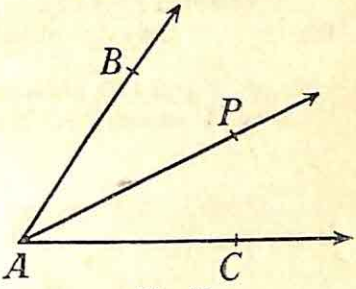


Fig. 53.

The Protractor Postulate. If $\overrightarrow{VA}$ is any ray lying in the edge of half plane H_1, there is one-to-one correspondence between the set of real numbers r where $0 \leqslant r \leqslant 180$ and the set of rays $\overrightarrow{VX}$ which lie in the union of H_1 and its edge such that :

(1) $\overrightarrow{VA}$ corresponds to the number zero.

(2) The ray opposite to $\overrightarrow{VA}$ corresponds to 180.

(3) If X and Y are points belonging to H_1 and which are not collinear with V and if x, y are the real numbers corresponding to $\overrightarrow{VX}$ and $\overrightarrow{VY}$ respectively, then $m\angle XVY = |x-y|$.

It may, however, be noted that in the statement of our postulate we have included the ray $\overrightarrow{VA}$ and the ray opposite to $\overrightarrow{VA}$ and hence the inequality $0 \leqslant r \leqslant 180$. Of course the measure of any angle will be larger than zero but less than 180.

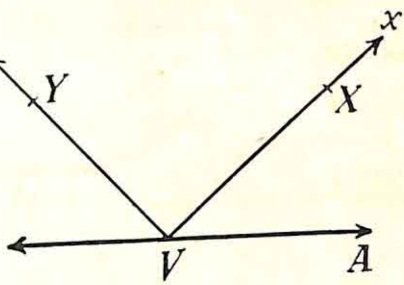


Fig. 54.

Now we prove the angle construction postulate as a theorem.

Theorem VI (1). (*Angle Construction Theorem*). If $\overrightarrow{VA}$ is any ray lying in the edge of half plane H_1 and if r is any number lying between 0 and 180, then there is a unique ray $\overrightarrow{VX}$ such that X belongs to H_1 and $m\angle AVX = r$.

Given : $\overrightarrow{VA}$ any ray lying in the edge of a half plane H_1 and r is any number such that $0 < r < 180$.

To prove that : There is a unique ray $\overrightarrow{VX}$ $(X \in H_1)$ such that $m\angle AVX = r$.

Proof. Protractor postulate helps us to assert that $\overrightarrow{VA}$ corresponds to 0 and there exists a unique ray $\overrightarrow{VX}$ $(X \in H_1)$ such that $\overrightarrow{VX}$ corresponds to r where $0 < r < 180$. (As the protractor postulate says that there is one-to-one correspondence between the set of real numbers r where $0 \leqslant r \leqslant 180$ and the set of rays $\overrightarrow{VX}$ lying in the union of H_1 and its edge.)

Also $m\angle AVX = |r - 0| = r$ (Protractor Postulate).

Therefore, there are two possibilities before the reader :

(1) Either to state separate postulates for angle measurement, angle construction and angle addition ; or

(2) To state the protractor postulate and to prove angle construction postulate as a theorem.

Linear Pair. Although we have excluded 'straight angles' from the set of angles by the way we choose to define the angle, yet we may often have to consider figures like one shown in Fig. 55, which suggest the notion of 'straight angles'. To describe such figures, the mathematicians have introduced the concept of a linear pair.

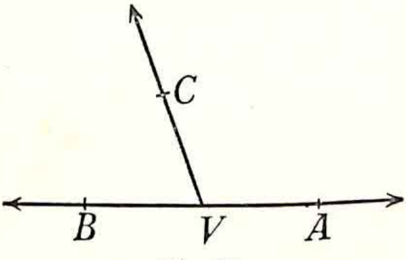
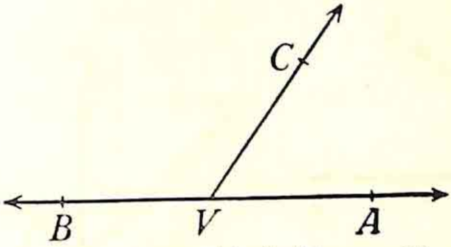


Fig. 55.

Definition VI (4). (*Linear Pair*). If $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are opposite rays and $\overrightarrow{VC}$ is any other ray then $\angle CVA$ and $\angle CVB$ are said to form a linear pair.

Adjacent Angles. Figs. 56, 57 and 58 are each a physical model of a pair of adjacent angles. We define adjacent angles as follows :



(Physical model of a Linear pair)
Fig. 56.

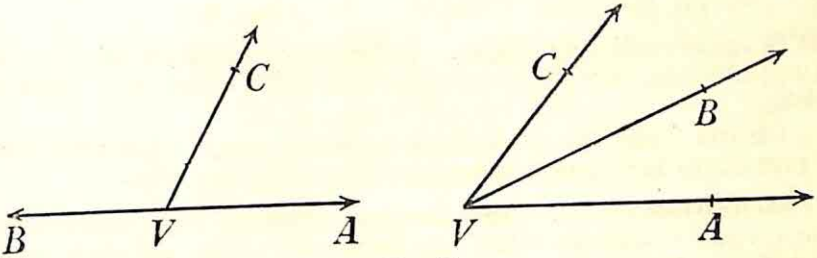


Fig. 57.

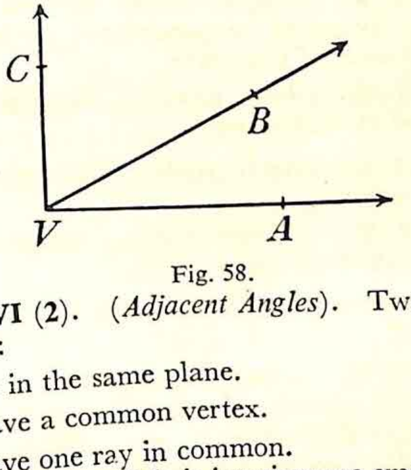


Fig. 58.

Definition VI (2). (*Adjacent Angles*). Two angles are called adjacent angles if :

(1) They lie in the same plane.
(2) They have a common vertex.
(3) They have one ray in common.
(4) The intersection of their interiors are empty.

The condition (1) has been imposed to exclude the possibility of two angles lying in different planes. The condition (2) has been imposed to exclude figures like the one given in Fig. 59 from the purview of adjacent angles.

The condition (3) has been imposed to exclude figures like the one given in Fig. 60 when it represents $\angle$BVA and $\angle$CVD. $\angle$BVA and $\angle$CVD are not adjacent angles.

The condition (4) has been imposed to exclude figures like one given in Fig. 60 when it represents $\angle$CVA and $\angle$BVD. $\angle$CVA and $\angle$BVD are not adjacent angles. In fact our definition of adjacent angles unambiguously states our common notion about the adjacent angles.

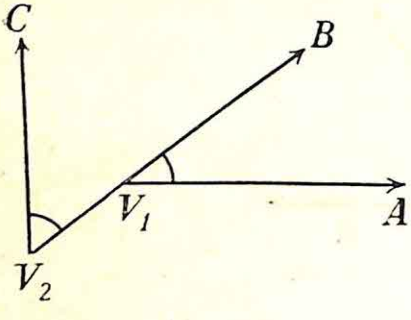


Fig. 59.

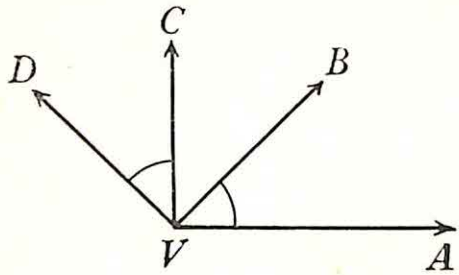


Fig. 60.

Clearly linear pair of angles is a special case of adjacent angles. We give below some more definitions pertaining to angles.

Definition VI (3). (*Supplementary Angles*). Two angles whose measures add to 180 are called supplementary angles and each angle is called the supplement of the other.

Definition VI (4). (*Complementary Angles*). Two angles whose measures add to 90 are called complementary angles and each angle is called the complement of the other.

Definition VI (5). (*Acute Angles*). An angle whose measure is less than 90 is called an acute angle.

Definition VI (6). (*Right Angle*). An angle whose measure is 90 is called a right angle.

Definition VI (7). (*Obtuse Angle*). An angle whose measure is more than 90 is called an obtuse angle.

Clearly the measure of an obtuse angle is more than 90 but less than 180.

Definition VI (8). (*Congruent Angles*). Two angles whose measures are equal are known as congruent angles. If $m\angle \text{ABC} = m\angle \text{DEF}$, then we can also write it as $\angle \text{ABC} \cong \angle \text{DEF}$. We will read it as : Angle ABC is congruent to angle DEF. Clearly all right angles are congruent.

Now congruency relation between angles is reflexive as any angle is congruent to itself. Also it is symmetric as if $\angle A \cong \angle B$, then $\angle B \cong \angle A$. Also it is transitive as if $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$.

Hence the congruency relation between the angles is an equivalence relation.

$\angle ABC = \angle DBF$ means that $\angle ABC$ is same as $\angle DBF$. Equality relation between two angles means something more than that their measures are equal. In fact it means that the two are the different names for the same angle. Since an angle is a set of points ; hence $\angle ABC = \angle DBF$ if they consist of exactly same points. In such a situation the physical model will be as in Fig. 61.

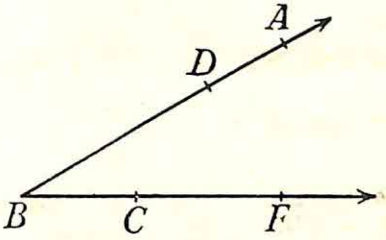


Fig. 61.

Obviously, equal angles are congruent but congruent angles may not be equal.

Now we are ready to take up some theorems of Euclidean geometry that have traditionally formed a part of curriculum of school geometry.

Theorem VI (1). The two angles of a linear pair are supplementary and the converse.

This theorem is generally taken up as a postulate as it is difficult to build up a very satisfactory proof on the basis of postulates we have stated so far. Generally for an elementary course, it is taken as a postulate which is stated below :

Postulate 15. (*The Supplement Postulate*). If two angles form a linear pair then they are supplementary.

However, somewhat satisfactory proof of this theorem can be built up in the following manner.

Given : $\overrightarrow{VA}$ and $\overrightarrow{VB}$, two opposite rays and $\overrightarrow{VC}$ another ray, forming a linear pair of angles consisting of $\angle CVA$ and $\angle CVB$.

To prove that : $m\angle CVA + m\angle CVB = 180$.

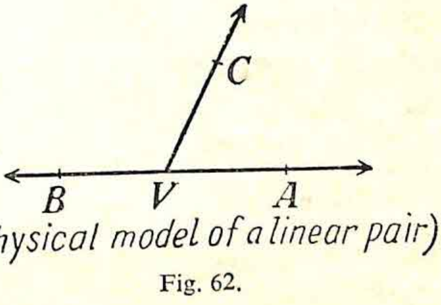


Fig. 62.

Proof.

Statement	*Reason*
(1) $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are opposite rays	(1) Given (premise)
(2) If $\overrightarrow{VA}$ corresponds to 0 then $\overrightarrow{VB}$ corresponds to 180	(2) Protractor postulate
(3) $m\angle CVA + m\angle CVB = \mid 180-0 \mid = 180$	(3) Protractor postulate.

By a similar reasoning, it can be proved that two angles forming a linear pair will be supplementary if $\overrightarrow{VA}$ and $\overrightarrow{VC}$ are the opposite rays and $\overrightarrow{VB}$ is any other ray or if $\overrightarrow{VB}$ and $\overrightarrow{VC}$ are opposite rays and $\overrightarrow{VA}$ is any other ray.

Converse of theorem I.

Statement. Supplementary adjacent angles form a linear pair.

Given : Two adjacent angles $\angle CVA$ and $\angle CVB$ such that $m\angle CVA + m\angle CVB = 180$.

To prove that : $\angle CVA$ and $\angle CVB$ form a linear pair.

Proof.

Statement	*Reason*
(1) $\angle CVA$ and $\angle CVB$ are adjacent angles such that : (*a*) V is the common vertex (*b*) $\overrightarrow{VC}$ is the common ray (*c*) $\angle CVA \cap \angle CVB = \varphi$ (*d*) $m\angle CVA + m\angle CVB = 180$	(1) Given
(2) Since $\overrightarrow{VC}$ is the common ray if $\overrightarrow{VA}$ corresponds to 0 then $\overrightarrow{VB}$ corresponds to 180	(2) Statement 1

(3) $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are opposite rays	(3) Statement 2, Protractor Postulate
(4) Since $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are opposite rays and $\overrightarrow{VC}$ is any other ray, $\angle CVA$ and $\angle CVB$ form a linear pair	(4) Definition of a linear pair.

Following are the riders to this theorem or Supplement Postulate.

Theorem VI (1A). If two angles of a linear pair are congruent, each is a right angle.

Given : Two adjacent angles $\angle CVA$ and $\angle CVB$ such that : $m\angle CVA = m\angle CVB$

or $\angle CVA \cong \angle CVB$

and $m\angle CVA + m\angle CVB = 180$.

To prove that : $m\angle CVA = m\angle CVB = 90$.

Proof.

Statement	*Reason*
(1) $m\angle CVA + m\angle CVB = 180$	(1) Given
(2) $m\angle CVA = m\angle CVB$	(2) Given
(3) $m\angle CVB + m\angle CVB = 180$ Or, $m\angle CVB = 90$	(3) Statements 1 and 2 on substituting the value of $m\angle CVA$
(4) $m\angle CVA = m\angle CVB = 90$	(4) Statements 2 and 3.

Theorem VI (1B). Supplements of the congruent angles are congruent.

Theorem VI (1C). Complements of congruent angles are congruent.

The proof of these two riders is left to the reader.

Vertical pair. In this figure $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have a point O in common. $\angle AOD$ and $\angle BOC$ are said to form a vertical pair. Similarly $\angle AOC$ and $\angle BOD$ form a vertical pair. So when two lines

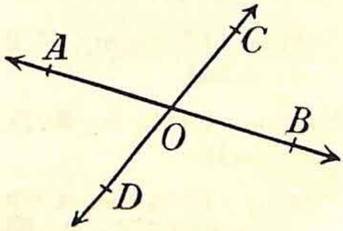


Fig. 63.

have a point in common, two pairs of vertical angles are formed. We define a vertical pair as follows :

Definition VI (9). (*Vertical Pair or a Pair of Vertical Angles*). Two angles are said to form a vertical pair if their sides form a pair of opposite rays.

Theorem VI (2). (*The Vertical Angle Theorem*). Two angles that form a vertical pair are congruent or vertical angles are congruent.

Given : Two pairs of opposite rays : $\overrightarrow{VA}$ opposite to $\overrightarrow{VB}$ and $\overrightarrow{VC}$ opposite to $\overrightarrow{VD}$ such that :

(1) $\angle CVA$, $\angle BVD$ are a pair of vertical angles.

(2) $\angle CVB$, $\angle AVD$ are another pair of vertical angles.

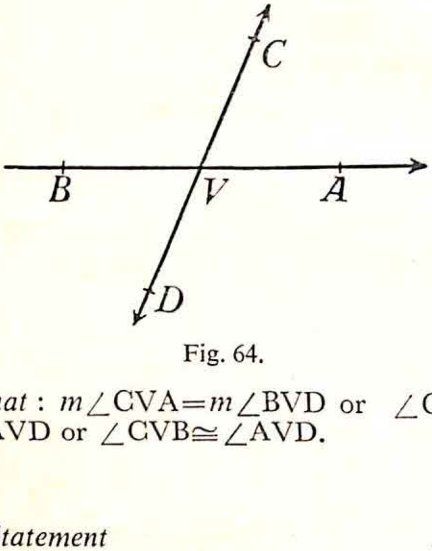


Fig. 64.

To prove that : $m\angle CVA = m\angle BVD$ or $\angle CVA \cong \angle BVD$ and $m\angle CVB = m\angle AVD$ or $\angle CVB \cong \angle AVD$.

Proof.

Statement	*Reason*
(1) $\overrightarrow{VB}$ is opposite to $\overrightarrow{VA}$ and $\overrightarrow{VC}$ is any other ray	(1) Given
(2) $m\angle CVA + m\angle CVB = 180$	(2) Statement 1 ; $\angle CVA$ and $\angle CVB$ form a linear pair
(3) $m\angle CVB + m\angle BVD = 180$	(3) $\angle CVB$ and $\angle BVD$ form a linear pair
(4) $m\angle CVA + m\angle CVB = m\angle CVB + m\angle BVD$	(4) Statements 2 and 3

(5) $m\angle CVA = m\angle BVD$ or $\angle CVA \cong \angle BVD$	(5) Statement 4.

Similarly, we can prove that $\angle CVB \cong \angle AVD$.

Theorem VI (2A). If $\overleftrightarrow{AB}$ is any line dividing a plane P into two half planes H_1 and H_2 and V is any point in $\overleftrightarrow{AB}$ and if $\overrightarrow{VC}$ is any ray in H_1 and $\overrightarrow{VD}$ is any ray in H_2 such that $\angle CVA \cong \angle BVD$, then $\angle CVA$, $\angle BVD$ form a vertical pair.

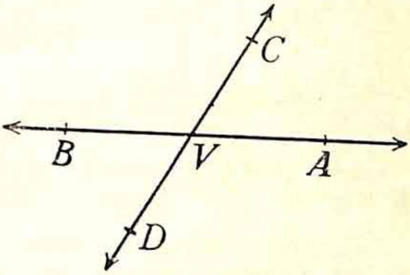


Fig. 65.

Given : $\overleftrightarrow{AB}$ divides a plane P into two half planes H_1 and H_2. $\overrightarrow{VC}$ is a ray in H_1 and $\overrightarrow{VD}$ is a ray in H_2 such that $\angle CVA \cong \angle BVD$.

To prove that : $\angle CVA$, $\angle BVD$ form a vertical pair.

Proof.

Statement	*Reason*
(1) $\overrightarrow{VA}$ is a ray opposite to $\overrightarrow{VB}$ and $\overrightarrow{VC}$ is any other ray	(1) $\overleftrightarrow{AB}$ is a line and V is any point in it.
(2) $m\angle CVA + m\angle CVB = 180$	(2) Statement 1, theorem VI (1)
(3) $\overrightarrow{VA}$ is a ray opposite to $\overrightarrow{VB}$ and $\overrightarrow{VD}$ is any other ray	(3) Why ?
(4) $m\angle AVD + m\angle BVD = 180$	(4) Why ?
(5) $m\angle CVA + m\angle CVB = m\angle AVD + m\angle BVD$	(5) Statements 2 and 4
(6) $m\angle CVA = m\angle BVD$	(6) Given
(7) $m\angle CVB = m\angle AVD$	(7) Statements 5 and 6
(8) $m\angle CVA + m\angle AVD = 180$	(8) Statements 2 and 7
(9) $\angle CVA$, $\angle AVD$ are such that (*a*) They lie in the same plane (*b*) They have the common vertex *d*V (*c*) $\angle CVA \cap \angle AVD = \varphi$	(9) Given

(10) ∠CVA, ∠AVD are adjacent supplementary angles	(10) Statements 8 and 9
(11) $\overrightarrow{VC}$ is opposite to $\overrightarrow{VD}$	(11) Statement 10 [Converse to theorem VI(1)].

Hence $\overrightarrow{VC}$ is opposite to $\overrightarrow{VD}$ and $\overrightarrow{VA}$ is opposite to $\overrightarrow{VB}$ (Why ?).

Therefore, ∠CVA, ∠BVD form a vertical angle.

Problem Set

1. Determine the measure of complementary angles of each of the angles whose measures are as follows :

(*a*) 73 (*b*) 34·5 (*c*) x (*d*) $(90-x)$ (*e*) $(90+x)$ (*f*) $(180-x)$ (*g*) 0·01 (*h*) 45 (*i*) 27·001 (*j*) 89·999.

2. Determine the measure of supplementary angles of each of the angles whose measures are given in problem 1.

3. If two angles with the same measure are complementary, then what is the measure of each ?

4. If two angles with the same measure are supplementary, then what is the measure of each ?

5. Two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have a point V in common, forming two pairs of vertical angles.

(*a*) If measure of $\angle AVB = 50$, then what is the measure of each of the remaining angles of the vertical pairs ?

(*b*) If all the angles of the vertical pairs are congruent to each other then what is the measure of each ?

6. The measure of an angle is five times the measure of its supplement. Find the measure of each.

7. Find the measure of the angle whose supplement is three times as large as its complement.

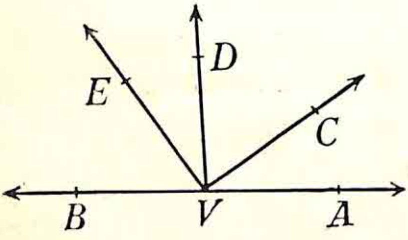


Fig. 66.

8. In the edge of a half plane H_1, $\overrightarrow{VA}$, $\overrightarrow{VB}$ are the opposite rays. $\overrightarrow{VC}$, $\overrightarrow{VD}$, $\overrightarrow{VE}$ are rays in H_1 such that $m\angle EVB = m\angle EVD$ and $m\angle DVC = m\angle CVA$. Find $m\angle EVC$.

9. $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$ are three distinct rays in a plane P such that

no two of them are opposite. Then, will :

(*a*) $m\angle AVB + m\angle BVC = m\angle AVC$?

(*b*) $m\angle AVB + m\angle BVC + m\angle AVC = 360$?

10. One of the two supplementary angles has a measure of 40 more than the measure of the other. What is the measure of each ?

11. The measure of an angle is two times the measure of its complement. What is the measure of each ?

12. Prove that if two angles are complementary, then each of them is acute.

13. Prove that if two intersecting lines form one right angle, then they form four right angles.

14. Prove that the bisectors of two supplementary adjacent angles are perpendicular to each other.

15. Prove that if two angles are right angles then they are congruent. Write the converse, opposite and contrapositive of this statement. Which of these will be true ?

16. Given two pairs of rays $\overrightarrow{VA}$ opposite to $\overrightarrow{VB}$ and $\overrightarrow{VC}$ opposite to $\overrightarrow{VD}$. If there is a ray $\overrightarrow{VE}$ between $\overrightarrow{VA}$ and $\overrightarrow{VC}$, then the ray $\overrightarrow{VF}$ opposite to $\overrightarrow{VE}$ is between $\overrightarrow{VB}$ and $\overrightarrow{VD}$.

We have already defined a right angle as an angle whose measure is 90. This concept of right angle is very closely connected to the concept of perpendicularity. Two rays that form a right angle are called perpendicular rays and are said to be perpendicular to each other. The lines determined by these rays will be called perpendicular lines. Segments determined by these rays are called perpendicular segments. A ray and a line can be perpendicular to each other. A line and a segment can be perpendicular to each other and so on. Hence we state the following definition of perpendicular lines and perpendiculars :

Definition VI (10). The lines determined by two rays which form a right angle are called perpendicular lines and any two sets, each of which may be a segment, a ray or a line and which determine two perpendicular lines are said to be perpendiculars.

If S_1 and S_2 are any two sets which are perpendicular to each other, then we write this as $S_1 \perp S_2$. Its physical model will be :

Two perpendicular sets may intersect each other as in the case of Fig. 67 or may not do so (Fig. 68). If they intersect then they will have at the most one point in common. If F is a point common between the perpendicular sets and if P is any other point in one of

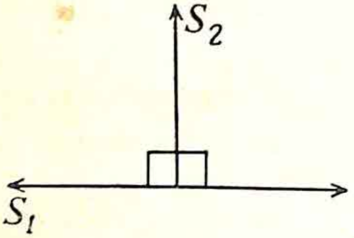


Fig. 67.

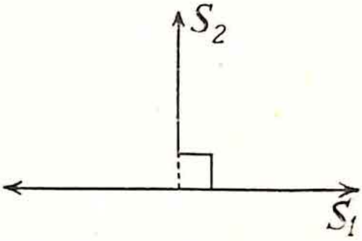


Fig. 68.

the sets but which does not belong to the second set, then F is said to be the foot of the perpendicular from P to the set that does not contain P. A line, a ray or a segment is a set of points and at each point we can have only one set of points which will be perpendicular to the given set of points. This is of fundamental importance and we prove it as a theorem.

Theorem VI (3). At each point of a given line, there is one and only one line which is perpendicular to the given line and lies in the plane which contains the given line.

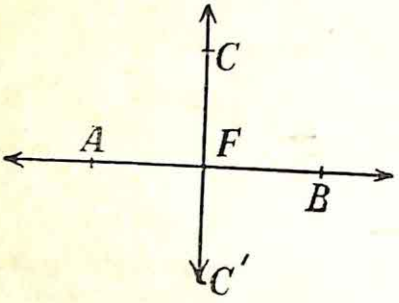


Fig. 69.

Given : A line $\overleftrightarrow{AB}$ lying in a plane P and F any point on $\overleftrightarrow{AB}$ (or $F \in \overleftrightarrow{AB}$).

To prove that : (*i*) There is at least one line in the plane P which is perpendicular to the given line at F.

(*ii*) There is atmost one line in the plane P which is perpendicular to the given line at F.

Proof.

Statement	*Reason*
(1) The line $\overleftrightarrow{AB}$ divides the plane P into two half planes H_1 and H_2	(1) Plane Separation Postulate
(2) There exists a point C in H_1 such that $\angle CFB$ is a right angle and the ray $\overrightarrow{FC}$ is unique	(2) Angle Construction theorem

(3) $\overleftrightarrow{FC} \perp \overleftrightarrow{AB}$	(3) Statement 2
(4) There exists at least one line in plane P which is perpendicular to the given line at the point F	(4) Statement 3
(5) There exists a point C′ in H_2 such that ∠C′FB is a right angle and the ray $\overrightarrow{FC'}$ is unique	(5) Angle Construction theorem
(6) $\overrightarrow{FC'} \perp \overleftrightarrow{AB}$	(6) Statement 5
(7) $m\angle CFB + m\angle C'FB = 180$	(7) Statements 3 and 6
(8) ∠CFB and ∠C′FB form a linear pair	(8) Statement 7
(9) $\overrightarrow{FC}$ and F′C are opposite rays	(9) Statement 8
(10) $\overleftrightarrow{FC}$ and $\overleftrightarrow{FC'}$ are the same line (or $\overleftrightarrow{FC} = \overleftrightarrow{FC'}$)	(10) Statement 9
(11) There is atmost one line in plane P which is perpendicular to the given line at F	(11) Statement 10.

We prove below some Incidence theorems based on the plane separation postulate and interior of an angle. We shall be using the results of these theorems in proving other theorems of the system.

Theorem VI (4). If A, B, C are non-collinear points and if A–C–D ; B–F–C and A–F–G then G is in the interior of ∠BCD, given that A, B, C, D, F, G are coplanar.

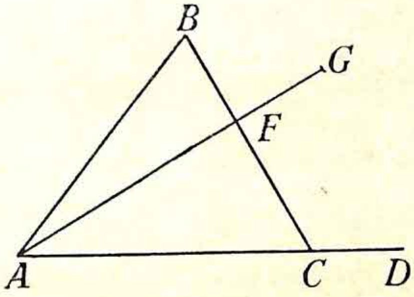


Fig. 70.

Given : (1) A, B, C three non collinear points.

(2) A – C – D ; B – F – C ; A – F – G.

(A – C – D means that C is between A and D.)

To prove that : G is in the interior of ∠BCD.

Proof.

Statement	*Reason*
(1) B and F are on the same side of $\overleftrightarrow{AC}$	(1) As B – F – C, $\overline{BF}$ does not intersect $\overleftrightarrow{AC}$.
(2) F, G are on the same side of $\overleftrightarrow{AC}$	(2) A – F – G ; hence $\overline{FG}$ does not intersect $\overleftrightarrow{AC}$
(3) B, G are on the same side of $\overleftrightarrow{AC}$	(3) Statements 1 and 2
(4) B, G are on the same side of $\overleftrightarrow{CD}$	(4) As $\overleftrightarrow{CD} = \overleftrightarrow{AC}$
(5) G lies on B-side of $\overleftrightarrow{CD}$	(5) Statement 4
(6) A and D are on opposite side of $\overleftrightarrow{BC}$	(6) A – C – D and as AD and $\overline{BC}$ intersect in C
(7) A and G are on opposite side of BC	(7) A – F – G and as $\overline{AG}$ and $\overline{BC}$ have F in common (B – F – C)
(8) G, D are on same side of $\overleftrightarrow{BC}$	(8) Statements 6 and 7 and theorem V (4)
(9) G lies on D-side of $\overleftrightarrow{BC}$	(9) Statements 6, 7, 8
(10) G lies on B-side of $\overleftrightarrow{CD}$ and D-side of $\overleftrightarrow{BC}$	(10) Statements 5 and 9
(11) G lies in the interior of ∠BCD	(11) Statement 10.

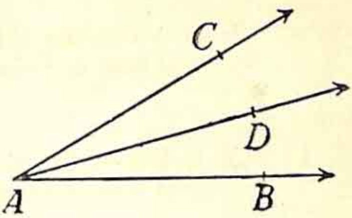


Fig. 71.

Theorem VI (5). If D lies in the interior of $\angle BAC$ then $\overrightarrow{AD}-A$ or $\overset{o\rightarrow}{AD}$ (all the points of the ray $\overrightarrow{AD}$ excepting its end point A) lies in the interior of $\angle BAC$.

Given : D lies in the interior of $\angle BAC$.

To prove that : $\overrightarrow{AD}-A$ lies in the interior of $\angle BAC$.

Proof.

Statement	*Reason*
(1) D lies on C-side of $\overrightarrow{AB}$ and B-side of $\overrightarrow{AC}$	(1) Given, definition of interior of an angle
(2) $\overrightarrow{AD}-A$, lies on B-side of $\overleftrightarrow{AC}$ and C-side of $\overleftrightarrow{AB}$	(2) Theorem V (5)
(3) $\overrightarrow{AD}-A$ Or $\overset{o\rightarrow}{AD}$ lies in the interior of $\angle CAB$	(3) Statement 2.

Theorem VI (6). If D lies in the interior of $\angle BAC$ and G–A–D then $\overset{o\rightarrow}{AG}$ ($\overrightarrow{AG}-A$) lies in the side of $\overleftrightarrow{AC}$ that does not contain B.

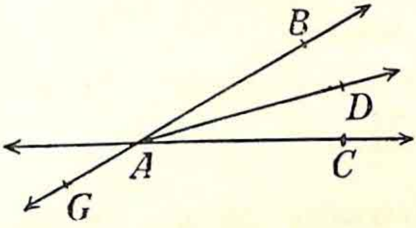


Fig. 72.

Proof.

Statement	*Reason*
(1) D lies on the B-side of $\overrightarrow{AC}$	(1) Given ; D lies in the interior of $\angle BAC$

(2) G lies on the side of $\overrightarrow{AC}$ that does not contain D	(2) (*a*) G—A—D (Given) (*b*) Plane Separation Postulate
(3) G lies on the side of $\overrightarrow{AC}$ that does not contain B	(3) Statements 1 and 2; since B and D lie on the same side of $\overrightarrow{AC}$
(4) All points of $\overrightarrow{AG}$—A lie on the same side of $\overrightarrow{AC}$	(4) Theorem V(5)
(5) $\overrightarrow{AG}$—A lie on the side of $\overleftrightarrow{AC}$ that does not contain B	(5) Statements 3 and 4.

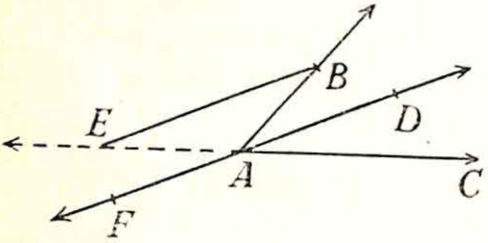


Fig. 73.

Theorem VI (7). If D is in the interior of ∠BAC and E—A—C then E and B are on the same side of $\overleftrightarrow{AD}$.

Given : (1) D is in the interior of ∠BAC.

(2) E—A—C.

To prove that : E and B are on the same side of $\overleftrightarrow{AD}$.

Proof.

Statement	*Reason*
(1) E and C are on opposite side of $\overleftrightarrow{AB}$	(1) E—A—C (Given)
(2) $\overrightarrow{BE}$—B lies on the side of $\overleftrightarrow{AB}$ that does not contain C	(2) Theorem V (5) and statement 1
(3) $\overline{EB}$ does not contain any point lying in the interior of ∠BAC	(3) Statement 2
(4) $\overline{EB}$ does not contain A	(4) E—A—C (Given)

(5) $\overline{EB}$ does not contain any point of $\overline{AD}$	(5) Since it does not either contain A or any point lying in the interior of $\angle BAC$ (Statements 3 and 4)
(6) $\overrightarrow{EB}-E$ lies on B-side of $\overleftrightarrow{AC}$	(6) Theorem V (5)
(7) $\overline{EB}-E$ lies on B-side of $\overleftrightarrow{AC}$	(7) Statement 6 ; since $\overline{EB} \subset \overrightarrow{EB}$
(8) If $F-A-D$ then $\overrightarrow{AF}-A$ lies on the side of $\overleftrightarrow{AC}$ that does not contain B	(8) Theorem VI (6)
(9) $\overrightarrow{AF}-A$ and $\overline{EB}-E$ lie on opposite side of $\overleftrightarrow{AC}$	(9) Statements 7 and 8
(10) $\overrightarrow{AF}$ does not contain E	(10) $E-A-C$ and $F-A-D$
(11) $\overrightarrow{AF}$ and $\overline{EB}$ do not have any point in common	(11) Statements 4, 9 and 10
(12) $\overline{EB}$ does not contain any point of $\overrightarrow{AF} \cup \overline{AD}$	(12) Statements 5 and 11
(13) $\overline{EB}$ does not contain any point of $\overleftrightarrow{AD}$	(13) Statement 12
(14) E and B are on the same side of $\overleftrightarrow{AD}$	(14) Statement 13.

Theorem VI (**8**). (*The Crossbar Theorem*). If D is in the interior of $\angle BAC$ then $\overrightarrow{AD}$ intersects $\overline{BC}$.

Given : D is in the interior of $\angle BAC$.

To prove that : $\overrightarrow{AD}$ intersects $\overline{BC}$.

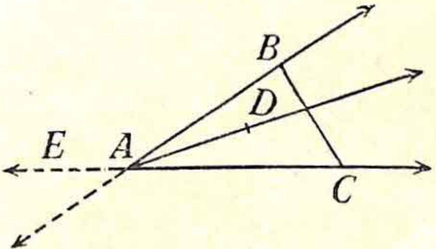


Fig. 74.

Proof.

We will prove it indirectly by contradiction. Assume that $\overrightarrow{AD}$ does not intersect $\overline{BC}$. Then B and C are on the same side of $\overleftrightarrow{AD}$. (Plane Separation Postulate).

In other words C lies on B-side of $\overleftrightarrow{AD}$. Also if E—A—C then E, B lie on same side of $\overleftrightarrow{AD}$ or E lies on B-side of $\overleftrightarrow{AD}$ [Theorem VII (7)]. It implies that E and C lie on B-side if $\overleftrightarrow{AD}$ or E and C lie on the same side of $\overleftrightarrow{AD}$ or $\overline{EC}$ does not have any point common with $\overleftrightarrow{AD}$ but this contradicts the statement that E—A—C. Hence the assumption that $\overrightarrow{AD}$ does not intersect $\overline{BC}$ has to be abandoned.

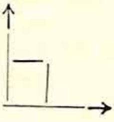
CHAPTER VII

TRIANGLES AND THEIR CONGRUENCE

The Triangle is most commonly encountered figure in elementary geometry. If A, B and C are any three non-collinear points, then the union of segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ is called a triangle. It is denoted by the symbol $\triangle ABC$. We can define a triangle as follows :

Definition VII (1). (*Triangle*). The union of three segments determined by three non-collinear points is called a triangle.

The points A, B, C of the $\triangle ABC$ are called its vertices and three segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are called its sides. The angles $\angle ABC$, $\angle BCA$ and $\angle CAB$ are called the angles of $\triangle ABC$. If one angle of a triangle is a right angle then it is called a **right triangle**. A triangle having the measure of its two sides equal (or with its two sides congruent) is called an **Isosceles triangle.** A $\triangle ABC$ will be isosceles if either $\overline{AB} \cong \overline{BC}$ or $\overline{BC} \cong \overline{AC}$ or $\overline{AB} \cong \overline{AC}$. If three sides of a triangle are congruent in pairs, the triangle is said to be **equilateral.** A $\triangle ABC$ will be equilateral if $\overline{AB} \cong \overline{BC}$, $\overline{BC} \cong \overline{AC}$ and $\overline{AC} \cong \overline{AB}$. Any side of a triangle and an angle whose vertex is not a point of that side are said to be *opposite* to each other. If all the angles of a triangle are equal in measure, then the triangle is called **equiangular** triangle.

In a right triangle, the side opposite to the vertex of right angle is called the hypotenuse and other two sides are called legs. The right angle is always indicated as . For any tri-

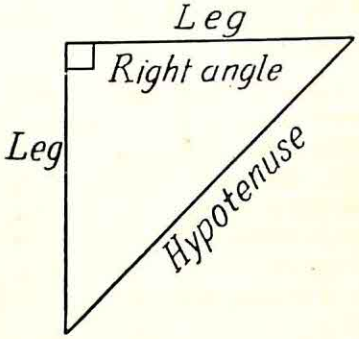


Fig. 75.

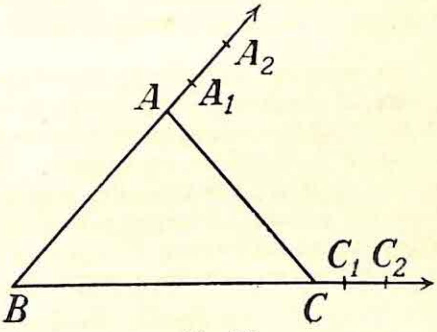


Fig. 76.

angle, the perpendicular segment from any vertex to the line determined by the opposite side is called the *altitude*. We have already stated that a $\triangle ABC$ determines three angles. It will be a proper question to ask whether the triangle contains its angles or not. We note that a triangle is a set of points and also an angle is a set of points. Now $\angle ABC$ is an angle of triangle ABC. Does $\triangle ABC$ contain $\angle ABC$? Obviously not because $\angle ABC$ contains points like A_1, A_2...... and points C_1, C_2...... which do not belong to the triangle. The $\triangle ABC$ does not contain points like A_1, A_2...... and C_1, C_2...... Therefore, although it is proper to speak of angle of a triangle or the angle determined by a triangle but it is incorrect to speak about 'the angle contained in a triangle'.

The classification of a triangle on the basis of the measure of its sides is as follows :

(1) A *Scalene triangle* is a triangle having no two sides equal in measure.

(2) An *Isosceles triangle* is a triangle having two sides equal in measure.

(3) An *Equilateral triangle* is a triangle having three sides equal in measure in pairs.

Congruence of two geometric figures. Roughly speaking two geometric figures are said to be congruent if they are of exactly same shape or size. If two figures are congruent and if it is possible to move one figure onto other, then through some trial and error, the two figures will coincide. For example, if we have two triangles, $\triangle ABC$ and $\triangle DEF$ with $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$ and $\overline{BC} \cong \overline{EF}$ and if it is possible to move $\triangle ABC$ onto $\triangle DEF$, then both may fit each other exactly if we move the vertex A on the vertex D, vertex B on vertex E and vertex C on the vertex F. Please note the use of phrase '*if possible*' in 'if it is possible to move one figure onto the other'. Congruence is correlation of two regions of space point by point so that all corresponding distances and all corresponding angles are equal in measure. Congruence is an idea of one-to-one correlation or correspondence of points in space.

We were earlier talking about the moving of one figure onto the other, if possible. In the example of congruence of two triangles, we stated that two triangles will fit each other if A moves onto D, B moves onto E and C moves onto F. (If the order of 'moving' the vertices is changed, the triangles may not fit each other.) So we can say that the vertex A corresponds to vertex D, vertex B corresponds to vertex E and the vertex C corresponds to vertex F. This is a one-to-one correspondence and this can be represented as follows :

$$A \leftrightarrow D, \quad B \leftrightarrow E, \quad C \leftrightarrow F$$

There are five more possibilities of writing one-to-one correspondence (or $1-1$ correspondence) between the vertices of two triangles, $\triangle ABC$ and $\triangle DEF$, which are as follows :

$A \leftrightarrow D,\ B \leftrightarrow F,\ C \leftrightarrow E$;
$A \leftrightarrow E,\ B \leftrightarrow F,\ C \leftrightarrow D,\ A \leftrightarrow E,\ B \leftrightarrow D,\ C \leftrightarrow F$;
$A \leftrightarrow F,\ B \leftrightarrow D,\ C \leftrightarrow E$ and $A \leftrightarrow F,\ B \leftrightarrow E,\ C \leftrightarrow D$.

But if $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$ and $\overline{BC} \cong \overline{EF}$, then the triangles will fit each other only if $A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$ provided each of $\triangle ABC$, $\triangle DEF$ is scalene. The $1-1$ correspondence which makes the triangle coincide or 'fit each other' is called a congruence between two triangles. We can write this correspondence as :

$A \leftrightarrow D$
$B \leftrightarrow E$
and $C \leftrightarrow F$

or, we can simply write it as $ABC \leftrightarrow DEF$ which implies that $A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$ and if we are given that $1-1$ correspondence $ABC \leftrightarrow DEF$ is a congruence then we know that $\overline{AB} \cong \overline{DE}$; $\overline{BC} \cong \overline{EF}$; $\overline{CA} \cong \overline{DF}$, $\angle A \cong \angle D$; $\angle B \cong \angle E$ and $\angle C \cong \angle F$. This congruence can be shown graphically by indications on corresponding angles and sides as in the figures given here.

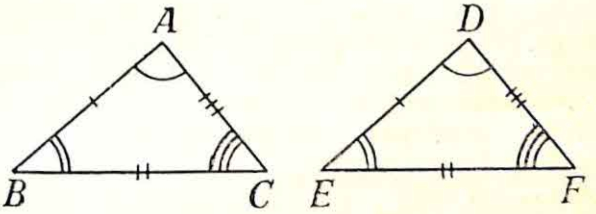


Fig. 77.

It may be noted that $1-1$ correspondence $ABC \leftrightarrow DEF$ is different from $1-1$ correspondence $ACB \leftrightarrow DEF$ for if we are given that $1-1$ correspondence $ACB \leftrightarrow DEF$ is a congruence then, $\angle A \cong \angle D$, $\angle C \cong \angle E$, $\angle B \cong \angle F$, $\overline{AC} \cong \overline{DE}$, $\overline{CB} \cong \overline{EF}$ and $\overline{AB} \cong \overline{DF}$ which is different from the implication of $1-1$ correspondence $ABC \leftrightarrow DEF$ being a congruence.

It is possible for two triangles to be congruent in more than one way. If we are given that each of the $\triangle ABC$ and $\triangle DEF$ is isosceles ($\overline{AB} \cong \overline{AC}$ and $\overline{DE} \cong \overline{DF}$) and $1-1$ correspondence $ABC \leftrightarrow DEF$ is a congruence and $\triangle ABC \cong \triangle DEF$ then the following $1-1$ correspondence will each be a congruence. (1) $ABC \leftrightarrow DEF$ and (2) $ABC \leftrightarrow DFE$. If we are given that each of the triangles, $\triangle ABC$ and $\triangle DEF$ is equilateral and $1-1$ correspondence $ABC \leftrightarrow DEF$ is a congruence and $\triangle ABC \cong \triangle DEF$ then the following $1-1$ correspondences will each be a congruence :

(1) $ABC \leftrightarrow DEF$ (2) $ABC \leftrightarrow EFD$ (3) $ABC \leftrightarrow FDE$
(4) $ABC \leftrightarrow DFE$ (5) $ABC \leftrightarrow EDF$ (6) $ABC \leftrightarrow FED$.

Now let us consider the case of only one triangle. When will a triangle (△ABC say) coincide with itself ? Obviously, 1—1 correspondence ABC↔ABC, will serve the purpose. In other words, every vertex would correspond with itself. Such a congruence is called *identity congruence*. If a △ABC is an isosceles triangle with $\overline{AB}\cong\overline{AC}$ then the following two 1—1 correspondences will give identity congruence. (1) ABC↔ABC (2) ABC↔ACB. If a △ABC is equilateral then the following six 1—1 correspondences will each give identity congruence. (1) ABC↔ABC (2) ABC↔ACB (3) ABC↔BCA (4) ABC↔BAC (5) ABC↔CAB (6) ABC↔CBA. We can define a congruence as follows :

Definition VII (2). (*A Congruence*). A 1—1 correspondence between the vertices of the same or different triangles such that the corresponding sides and corresponding angles are congruent is called a congruence between the triangles.

Definition VII (3). (*Congruent Triangles*). Two triangles are congruent iff (if and only if) there exists a congruence between them. Summarising, we state the following :

(1) Any triangle ABC is congruent to itself with 1—1 correspondence ABC↔ABC being a congruence.

(2) There are six possible 1—1 correspondences between the vertices of two triangles and in general not even one of these correspondences may be a congruence. But if it so happens that one of these correspondences is a congruence then the triangles are said to be congruent.

(3) According to our definition, before we can assert that two triangles are congruent, we must know that six simple congruence relations hold. These are : (1) three congruences between corresponding sides and (2) three congruences between corresponding angles.

(4) The definition of the concept of congruence in the form in which it has been given here is of much help in making the statements about the congruent triangles self explanatory and exact. If we are given that 1—1 correspondence ABC↔DEF is a congruence then we know that $\overline{AB}\cong\overline{DE}$, $\overline{BC}\cong\overline{EF}$, $\overline{AC}\cong\overline{DF}$; $\angle A\cong\angle D$; $\angle B\cong\angle E$ and $\angle C\cong\angle F$.

If a statement '△ABC is congruent to △DEF' occurs in a course of elementary geometry with conventional treatment in which the definition of congruent triangles has not been based on 1—1 correspondence in the form in which we have defined it then we will not know whether $\angle A\cong\angle D$ or $\angle A\cong\angle E$ or $\angle A\cong\angle F$ and so on. In fact this statement can stand for any one of the six possible 1—1 correspondences between the vertices of △ABC and △DEF. So the way we have defined the congruence between two triangles is a big improvement over the definition given in a conventional course in elementary geometry.

The statement $\triangle ABC \cong \triangle DEF$ in a modern course in elementary geometry implies that these triangles are congruent in a particular way *i.e.*, under $1-1$ correspondence $ABC \leftrightarrow DEF$. The statement $\triangle ABC \cong \triangle DFE$ will not be the same as the statement $\triangle ABC \cong \triangle DEF$.

We explained the concept of congruence between two triangles by the statement 'if it is possible to move one figure on to the other'. This approach was adopted simply to make the concept clear. The method of 'superposition of figures which is used as a method of proof for establishing congruence between two triangles in the conventional treatment of elementary geometry is defective and will not be used in this course. We know that if $1-1$ correspondence between the vertices of two triangles is a congruence then six simple congruence relations—three between the corresponding sides and three between corresponding angles hold. However, if certain sets of three of these six simple congruence relations hold, then the remaining three congruence relations will hold also and the triangles will be congruent. There are four such sets. In a conventional course of elementary geometry which uses the method of superposition of figures all these four cases are proved. However, in the present course, we will take one of these as a postulate and then we prove the remaining three. Hence we state the following postulate.

Postulate 16. (*S.A.S. Postulate*). If there exists $1-1$ correspondence between two triangles (or between a triangle and itself) in which two sides and the angle *determined* by these sides of one triangle are congruent to the corresponding parts of the second triangle, the correspondence is a congruence and the triangles are congruent. (S.A.S. 'stands for' Side-Angle-Side and $1-1$ correspondence between two triangles means that there exists a $1-1$ correspondence between the vertices of two triangles.)

In a course of elementary geometry with conventional treatment, this postulate is proved as a theorem by superposition of figures—'Place the triangle ABC over the triangle DEF such that......' which is defective. The postulates that we have stated earlier do not help us in proving this as a theorem. Hence we have stated it as a postulate.

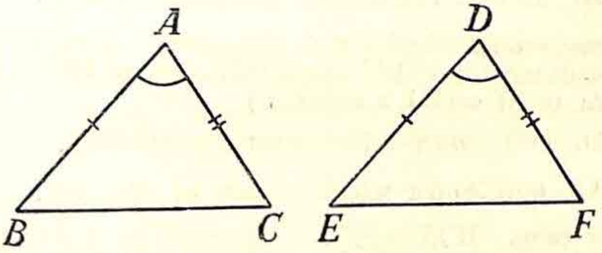


Fig. 78.

It may be noted that as stated in this postulate, the $1-1$ correspondence $ABC \leftrightarrow DEF$ will be a congruence iff $\overline{AB} \cong \overline{DE}$,

$\overline{AC} \cong \overline{DF}$ and $\angle A \cong \angle D$. Had we been given that $\angle B \cong \angle E$ or $\angle C \cong \angle F$ instead of having been given $\angle A \cong \angle D$ along with $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$ then the correspondence would not have been a congruence. The use of the phrase 'the angle determined by these sides' in the statement of this postulate may be noted in this connection.

With the help of S.A.S. postulate, we prove the following theorem :

Theorem VII (1). (*A.S.A. Theorem*). If there exists a $1-1$ correspondence between two triangles (or between a triangle and itself) in which two angles and the side common to two angles in one triangle are congruent to the corresponding parts of the other triangle, then the correspondence is a congruence and the triangles are congruent (A.S.A. 'stands for' Angle-Side-Angle).

Given : Two triangles ; $\triangle ABC$ and $\triangle DEF$ in which $A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$. Also $\angle B \cong \angle E$, $\angle C \cong \angle F$ and $\overline{BC} \cong \overline{EF}$.

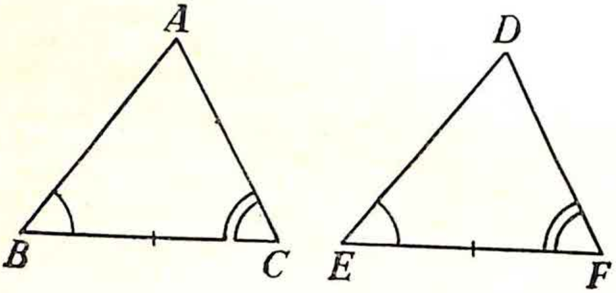


Fig. 79.

To prove that : $1-1$ correspondence $ABC \leftrightarrow DEF$ is a congruence and $\triangle ABC \cong \triangle DEF$.

Plan of Proof. We can apply S.A.S. Postulate, if in addition to what is given to us we can prove that either $\overline{AC} \cong \overline{DF}$ or $\overline{AB} \cong \overline{DE}$. Hence we will consider the measures of $\overline{AC}$ and $\overline{DF}$.

Proof. Law of Trichotomy tells us that either :

(1) $m\overline{AC} = m\overline{DF}$ which means the same as $\overline{AC} \cong \overline{DF}$ or $AC = DF$. (As AC means $m\overline{AC}$ and DF means $m\overline{DF}$. AC or DF each is a number while $\overline{AC}$ or $\overline{DF}$ each is a segment.)

(2) Or, $m\overline{AC} < m\overline{DF}$ which is same as $AC < DF$.

(3) Or, $m\overline{AC} > m\overline{DF}$ which is same as $AC > DF$.

First case. If $\overline{AC} \cong \overline{DF}$ then as we are given that $\overline{BC} \cong \overline{EF}$ and $\angle C \cong \angle F$ then $1-1$ correspondence $ABC \leftrightarrow DEF$ is a congruence and $\triangle ABC \cong \triangle DEF$ (S.A.S. Postulate).

Second case. When $m\overline{AC} < m\overline{DF}$.

Statement	*Reason*
(1) There exists a point D′ such that F−D′−D and such that $\overline{AC} \cong \overline{D'F}$	(1) Segment Construction Theorem [Theorem IV (5)]
(2) 1−1 correspondence ABC↔D′EF is a congruence and $\triangle ABC \cong \triangle D'EF$	(2) S.A.S. postulate
(3) $\angle CBA \cong \angle FED'$	(3) Corresponding parts of congruent triangles
(4) $\angle CBA \cong \angle FED$	(4) Hypothesis or given
(5) $\angle FED' \cong \angle FED$	(5) Statements 3 and 4
(6) D and D′ are on the same side of $\overleftrightarrow{EF}$	(6) F−D′−D (which means that F is not between D and D′)
(7) $\overrightarrow{ED} = \overrightarrow{ED'}$	(7) Statements 5 and 6; Angle Construction Postulate
(8) D=D′	(8) Statement 7 and as two lines intersect atmost one point.

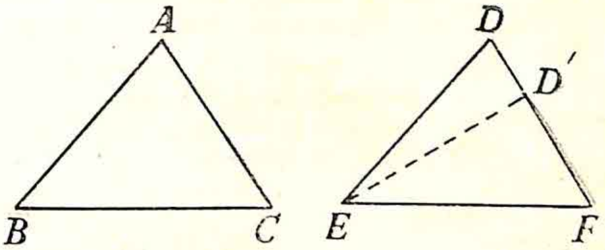


Fig. 80.

We have arrived at the result that D=D′ but this is impossible since D′ is between F and D. Hence the assumption that $m\overline{AC} < m\overline{DF}$ leads to a contradiction and this must be abandoned.

By proceeding on similar lines we can prove that the assumption taken in the third case leads to a contradiction and this will have to be abandoned. Hence if A.S.A. of one triangle are respectively congruent to A.S.A. of the other triangle then that particular 1−1 correspondence between the vertices of two triangles is a congruence and the triangles are congruent.

Now we prove some more theorems based on the concept of congruence of two triangles.

Theorem VII (2). (*Isosceles Triangle Theorem*). If two sides of a triangle are congruent, then the angles opposite these sides are also congruent.

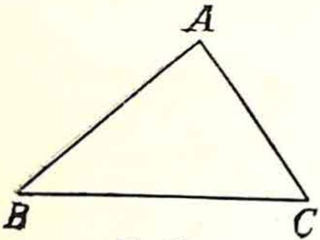


Fig. 81.

Given : $\triangle ABC$ in which $\overline{AB}\cong\overline{AC}$.

To prove that : $\angle C\cong\angle B$.

Proof. Let us consider the 1 – 1 correspondence $ABC\leftrightarrow ACB$ (or $A\leftrightarrow A$, $B\leftrightarrow C$ and $C\leftrightarrow B$) between the given triangle and itself (or between $\triangle ABC$ and $\triangle ACB$).

Statement	*Reason*
(1) $\overline{AB}\cong\overline{AC}$	(1) Given
(2) $\angle BAC\cong\angle CAB$	(2) Any angle is congruent to itself (or congruence between angles is an equilence relation)
(3) $\overline{AC}\cong\overline{AB}$	(3) Given ; congruence of segments is a symmetric relation
(4) 1 – 1 correspondence $ABC\leftrightarrow ACB$ is a congruence and $\triangle ABC\cong\triangle ACB$	(4) S.A.S. postulate
(5) $\angle ABC\cong\angle ACB$	(5) Corresponding parts of congruent triangles.

Theorem VII (3). (*Converse to Isosceles Triangle Theorem*). If two angles of a triangle are congruent, then the sides opposite these angles are also congruent.

Given : $\triangle ABC$ in which $\angle B\cong\angle C$.

To prove that : $\overline{AB}\cong\overline{AC}$.

Proof. Consider the 1 – 1 correspondence $ABC\leftrightarrow ACB$ between the triangle and itself (or between $\triangle ABC$ and $\triangle ACB$).

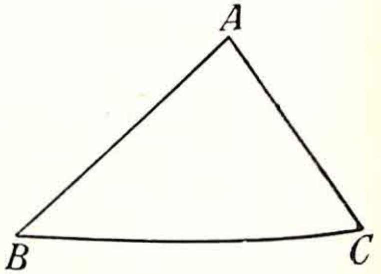


Fig. 82.

Statement	*Reason*
(1) $\angle B\cong\angle C$	(1) Given
(2) $\overline{BC}\cong\overline{CB}$	(2) A segment is congruent to itself
(3) $\angle C\cong\angle B$	(3) Given as congruence of angle is a symmetric relation

(4) 1 – 1 correspondence $ABC \leftrightarrow ACB$ is a congruence and $\triangle ABC \cong \triangle ACB$	(4) A.S.A. Theorem
(5) $\overline{AB} \cong \overline{AC}$	(5) Corresponding parts of congruent triangles.

Theorem VII (4). If in a $\triangle ABC$, $\overline{AB} \cong \overline{AC}$ then the segment determined by A and mid-point of $\overline{BC}$ is perpendicular to $\overline{BC}$.

Given : A $\triangle ABC$ in which $\overline{AB} \cong \overline{AC}$ and D is the mid-point of $\overline{BC}$.

To prove that : $\overline{AD} \perp \overline{BC}$.

Proof.

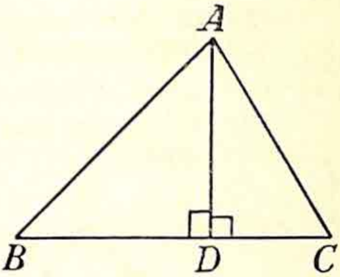


Fig. 83.

Statement	*Reason*
(1) $\overline{AB} \cong \overline{AC}$	(1) Given
(2) $\angle ABD \cong \angle ACD$	(2) Isosceles Triangle Theorem
(3) $\overline{BD} \cong \overline{CD}$	(3) Given that D is mid. point of $\overline{BC}$
(4) 1 – 1 correspondence $ABD \leftrightarrow ACD$ is a congruence and $\triangle ABD \cong \triangle ACD$	(4) S.A.S. Postulate
(5) $\angle ADB \cong \angle ADC$	(5) Corresponding parts of congruent triangles
(6) $\angle ADB$, $\angle ADC$ form a linear pair	(6) $\overrightarrow{DC}$ is opposite to $\overrightarrow{DB}$ and $\overrightarrow{DA}$ is any other ray
(7) Each of $\angle ADB$, $\angle ADC$ is a right angle	(7) Statements 5 and 6 [Theorem VI (1A)]
(8) $\overline{AD} \perp \overline{BC}$	(8) Statement 7.

Theorem VII (4). [*Converse to Theorem VII (3)*]. The perpendicular bisector of the base of an isosceles triangle passes through the vertex opposite that side.

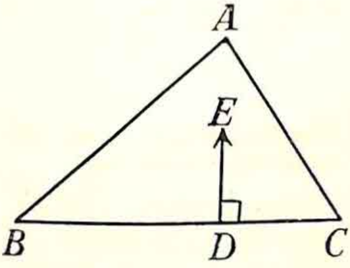


Fig. 84.

Given : A $\triangle ABC$ in which $\overline{AB} \cong \overline{AC}$ and $\overrightarrow{DE}$ is the perpendicular bisector of the base $\overline{BC}$.

To prove that : $\overrightarrow{DE}$ passes through the vertex A.

Proof. There are three probabilities in which $\overrightarrow{DE}$ can 'behave'.

(1) $\overrightarrow{DE}$ passes through A.

(2) $\overrightarrow{DE}$ does not pass through A but intersects $\overline{CA}$ in its interior point A′.

(3) $\overrightarrow{DE}$ does not pass through A but intersects $\overline{BA}$ in its interior point A″.

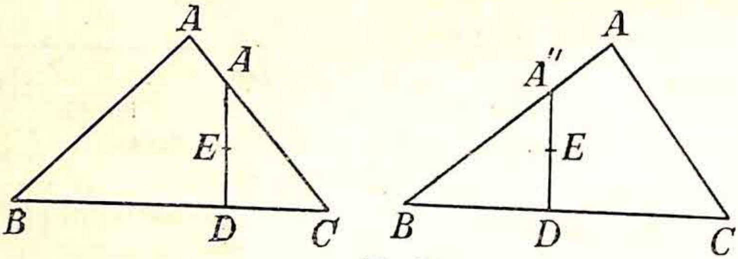


Fig. 85.

Case I : If $\overrightarrow{DE}$ passes through A, then there is nothing further to prove.

Case II : When $\overrightarrow{DE}$ intersects $\overline{CA}$ in its interior point A′. Consider the correspondence A′BD↔A′CD.

Statement	*Reason*
(1) $\overline{BD} \cong \overline{CD}$	(1) Given
(2) $\angle A'DB \cong \angle A'DC$	(2) Each is a right angle
(3) $\overline{A'D} \cong \overline{A'D}$	(3) A segment is congruent to itself
(4) 1–1 correspondence A′BD↔A′CD is a congruence and $\triangle A'BD \cong \triangle A'CD$	(4) S.A.S. postulate
(5) $\angle A'BD \cong \angle A'CD$	(5) Corresponding parts of congruent triangles
(6) $\angle ABD \cong \angle A'CD$	(6) Given, Theorem VII (2)
(7) $\angle A'BD \cong \angle ABD$	(7) Statements 5 and 6 as C—A′—A and hence $\angle A'CD = \angle ACD$
(8) $\angle A'BC \cong \angle ABC$	(8) Statement 7

(9) A and A′ are on the same side of $\overline{BC}$	(9) C is not between A and A′
(10) $\overrightarrow{BA}=\overrightarrow{BA'}$	(10) Angle Construction Postulate
(11) A=A′	(11) Two lines intersect in atmost one point.

We have arrived at the result that A=A′ which is impossible as A′ is between C and A. Hence the assumption that $\overrightarrow{DE}$ does not pass through A has to be abandoned.

By proceeding on similar lines we can prove that the third assumption that $\overrightarrow{DE}$ does not pass through A and instead intersects $\overline{BA}$ in its interior point A″ leads to a contradiction and hence this assumption must also be abandoned. Hence only the first assumption that $\overrightarrow{DE}$ passes through the vertex A holds.

Theorem VII (5). (*S.S.S. Theorem*). If there exists a 1−1 correspondence between two triangles (or between a triangle and itself) in which three sides of one triangle are congruent to the corresponding sides of the other triangle, the correspondence is a congruence and the triangles are congruent. (S.S.S. 'stand for' Side-Side-Side.)

Given: $\triangle ABC$ and $\triangle DEF$ such that A↔D, B↔E and C↔F and also $\overline{AB}\cong\overline{DE}$, $\overline{BC}\cong\overline{EF}$ and $\overline{AC}\cong\overline{DF}$.

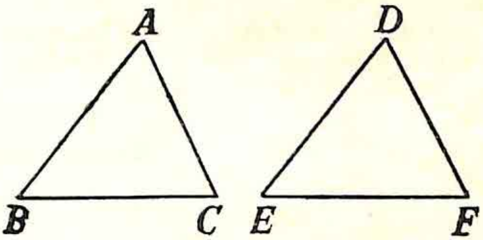


Fig. 86.

To prove that: 1−1 correspondence ABC↔DEF is a congruence and $\triangle ABC\cong\triangle DEF$.

Plan of Proof. If we can prove either one of the three simple congruences *viz.*, (1) $\angle ABC\cong\angle DEF$ (2) $\angle BCA\cong\angle EFD$ (3) $\angle CAB\cong\angle FDE$, then we can apply S.A.S. postulate and arrive at the required result. We will consider the measures of $\angle ABC$ and $\angle DEF$.

Proof. By Trichotomy law we know that:

(1) Either $\angle ABC \cong \angle DEF$

(2) Or, $m\angle ABC < m\angle DEF$

(3) Or, $m\angle ABC > m\angle DEF$.

Case 1. If $\angle ABC \cong \angle DEF$ then as we are given that $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, then by S.A.S. postulate we know that 1—1 correspondence $ABC \leftrightarrow DEF$ is a congruence and $\triangle ABC \cong \triangle DEF$.

Case 2. When $m\angle ABC < m\angle DEF$.

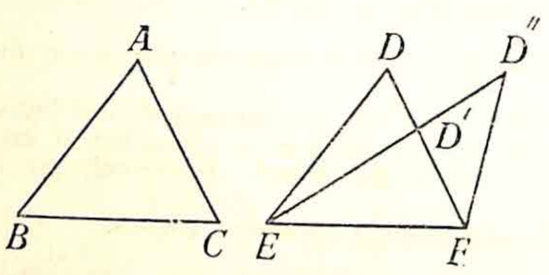


Fig. 87.

Statement	*Reason*
(1) There is a ray $\overrightarrow{ED'}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ such that $\angle D'EF \cong \angle ABC$	(1) Angle construction postulate and angle addition postulate
(2) On $\overrightarrow{ED'}$, there is a point D'' such that $\overline{ED''} \cong \overline{AB}$	(2) Segment Construction Theorem
(3) $\overline{AB} \cong \overline{ED''}$	(3) Statement 2
(4) $\angle ABC \cong \angle D''EF$	(4) Statement 1
(5) $\overline{BC} \cong \overline{EF}$	(5) Given
(6) 1—1 correspondence $ABC \leftrightarrow D''EF$ is a congruence and $\triangle ABC \cong \triangle D''EF$	(6) S.A.S. postulate
(7) $\overline{FD''} \cong \overline{CA}$	(7) Corresponding parts o congruent triangles
(8) $\overline{CA} \cong \overline{FD}$	(8) Given
(9) $\overline{FD''} \cong \overline{FD}$	(9) Statements 7 and 8
(10) $D''FD$ is an isosceles triangle	(10) Statement 9

(11) DED″ is an isosceles triangle	(11) $\overline{ED''} \cong \overline{AB}$ (Statement 2) and $\overline{AB} \cong \overline{ED}$ (Given)
(12) The perpendicular bisector of DD″ must pass through F	(12) Statement 10 and Theorem VII (4)
(13) The perpendicular bisector of DD″ must pass through E	(13) Statement 11
(14) $\overline{EF}$ is the perpendicular bisector of DD″	(14) Statements 12 and 13 ; Postulate 2
(15) D and D″ are on the same side of $\overleftrightarrow{EF}$	(15) $\overrightarrow{ED'}$ is between $\overrightarrow{ED}$ and $\overrightarrow{EF}$; Statement 1.

Now statements 14 and 15 contradict each other. (How ?). Hence the assumption that $m\angle ABC < m\angle DEF$ must be abandoned. By proceeding on similar lines we can prove that the third assumption that $m\angle ABC > m\angle DEF$ need to be abandoned as it leads to a similar contradiction. Hence the first assumption that $\angle ABC \cong \angle DEF$ holds.

Alternative Proof of S.S.S. Theorem.

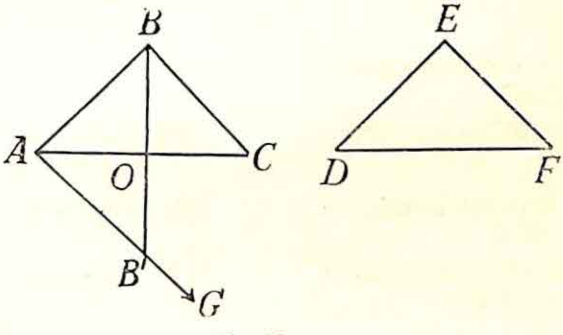


Fig. 88.

Proof.

Statement	*Reason*
(1) There exists a ray $\overrightarrow{AG}$ on the opposite of B-side of $\overleftrightarrow{AC}$ such that $\angle CAG \cong \angle EDF$	(1) Angle Construction Postulate
(2) There is a point B′ on $\overrightarrow{AG}$ such that $\overline{AB'} \cong \overline{DE}$	(2) Segment Construction Theorem

(3) $\overline{AC}\cong\overline{DF}$	(3) Given
(4) $1-1$ correspondence $AB'C \leftrightarrow DEF$ is a congruence and $\triangle AB'C\cong\triangle DEF$	(4) S.A.S. Postulate
(5) $\overline{B'C}\cong\overline{EF}$	(5) Corresponding parts of congruent triangles
(6) $\overline{BB'}$ intersects $\overline{AC}$ at O	(6) B and B′ are on opposite sides of $\overline{AC}$; Plane Separation Postulate.

Now depending upon whether the given triangles are acute, obtuse or right triangles, the order relation between A, O, C would be one of the following :

(*a*) $A-O-C$ (*b*) $O-A-C$ or $A-C-O$ (*c*) $O=C$ or $C=O$ or O coincides with C.

We will take up the first case $A-O-C$ and build the proof further.

(7) $\angle ABO\cong\angle AB'O$	(7) Isosceles Triangle Theorem $\overline{AB'}\cong\overline{DE}$ and $DE\cong\overline{AB}$
(8) $\angle CBO\cong\angle CB'O$	(8) Isosceles Triangle Theorem
(9) O is in the interior of $\angle ABC$	(9) $A-O-C$
(10) $m\angle ABC=m\angle ABO+m\angle CBO$	(10) Statement 9 ; the Angle Addition Postulate
(11) O is in the interior of $\angle AB'C$	(11) $A-O-C$
(12) $m\angle AB'C=m\angle AB'O+m\angle CB'O$	(12) Statement 11
(13) $\angle ABC\cong\angle AB'C$	(13) Statements 7, 8, 10, 12
(14) $\overline{AB}\cong\overline{AB'}$	(14) $\overline{AB'}=\overline{DE}$ (Statement 2) and $\overline{AB}=\overline{DE}$ (Given)
(15) $\overline{BC}\cong\overline{B'C}$	(15) $\overline{B'C}\cong\overline{EF}$ (Statement 5) $\overline{EF}\cong\overline{BC}$ (Given)
(16) $1-1$ correspondence $ABC\leftrightarrow AB'C$ is a congruence and $\triangle ABC\cong\triangle AB'C$	(16) S.A.S. Postulate
(17) $\triangle ABC\cong\triangle DEF$ and $1-1$ correspondence $ABC\leftrightarrow DEF$ is a congruence	(17) Statements 4 and 16.

Similarly, we can prove the cases b and c. The figures in these cases will be as follows :

(*Case B*)

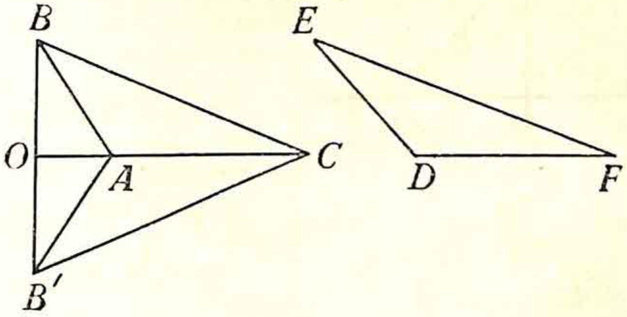


Fig. 89.

(*Case C*)

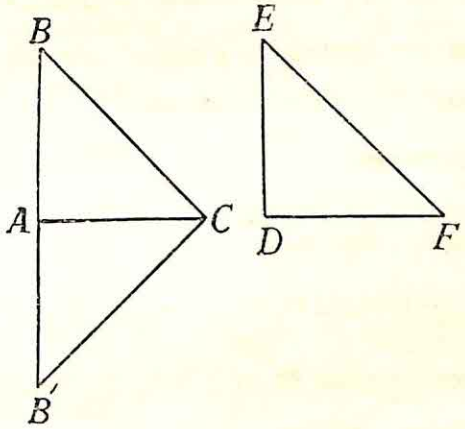


Fig. 90.

Theorem VII (6). (*Perpendicular from a point not on the line to the given line*). Given a line and a point not on the line, then there is one and only one line which passes through the given point and is perpendicular to the given line.

Given : A line $\overleftrightarrow{AB}$ and a point P such that $P \notin \overleftrightarrow{AB}$.

To prove that : (1) There is at least one line through P, perpendicular to $\overleftrightarrow{AB}$.

(2) There is atmost one line (not more than one) through P, perpendicular to $\overleftrightarrow{AB}$.

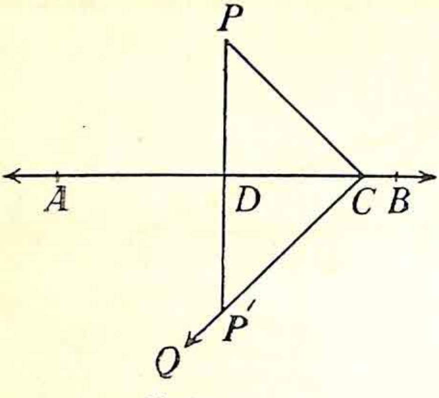


Fig 91.

Proof. Let C be a point such that $C \in \overleftrightarrow{AB}$ and A−C−B. Then there are only following three probabilities.

(1) $\overleftrightarrow{PC}$ will be perpendicular to $\overleftrightarrow{AB}$.

(2) $\angle ACP$ is an acute angle.

(3) $\angle ACP$ is an obtuse angle.

First Case. If $\overleftrightarrow{PC}$ is perpendicular to $\overleftrightarrow{AB}$ (or $\overleftrightarrow{PC} \perp \overleftrightarrow{AB}$) then we prove that there is atleast one line through $P \perp \overleftrightarrow{AB}$.

Second Case. If $\angle ACP$ is acute then :

Statement	*Reason*
(1) There exists a point Q on the opposite to the P-side of $\overleftrightarrow{AB}$ such that $\angle PCA \cong \angle ACQ$	(1) Angle Construction Postulate
(2) There exists a point P′ on $\overrightarrow{CQ}$ such that $\overline{CP} \cong \overline{CP'}$	(2) Segment Construction Theorem
(3) P′ is opposite to P-side of $\overleftrightarrow{AB}$	(3) Statement 1
(4) $\overleftrightarrow{PP'}$ intersects $\overleftrightarrow{AB}$ in D	(4) Plane Separation Postulate
(5) $\angle PCA \cong \angle ACQ$ $\overline{CP} \cong \overline{CP'}$ $\overline{CD} \cong \overline{CD}$	(5) Statements 1, 2 ; Any segment is congruent to itself
(6) 1−1 correspondence $DPC \leftrightarrow DP'C$ is a congruence and $\triangle DPC \cong \triangle DP'C$	(6) S.A.S. Postulate
(7) $\angle PDC \cong \angle P'DC$	(7) Statement 6

(8) $\overrightarrow{DP'}$ is opposite to $\overrightarrow{DP}$ and $\overrightarrow{DC}$ is any other ray	(8) $\overleftrightarrow{PP'}$ is a line and D is a point belonging to it
(9) ∠PDC, ∠P′DC form a linear pair	(9) Statement 8
(10) Each of ∠PDC, ∠P′DC is a right angle	(10) Steps 7, 9 [Theorem VI (1A)]
(11) $\overleftrightarrow{PP'} \perp \overleftrightarrow{AB}$	(11) Statement 10
(12) There exists atleast one line from P perpendicular to $\overleftrightarrow{AB}$	(12) Statement 11.

By taking the third case we can again prove (on similar lines as for second case) that there exists at least one perpendicular line from P to $\overleftrightarrow{AB}$.

Now we want to prove that there exists atmost one perpendicular line from P to $\overleftarrow{AB}$.

Let us assume that there exist two distinct perpendicular lines. $\overleftrightarrow{PD}$ and $\overleftrightarrow{PD'}$ from P to $\overleftrightarrow{AB}$.

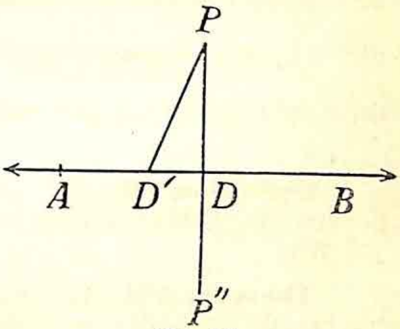


Fig. 92.

Statement	*Reason*
(1) There exists a point P″ on the opposite to P-side of $\overleftrightarrow{AB}$ such that $\overline{PD} \cong P''D$	(1) Segment Construction Theorem
(2) ∠PDD′≅∠P″DD′	(2) Each is a right angle (Given)
(3) $\overline{DD'} \cong \overline{DD'}$	(3) Any segment is congruent to itself
(4) 1−1 correspondence PDD′↔P″DD′ is a congruence and △PDD′≅△P″DD′	(4) S.A.S. Postulate
(5) ∠PD′D≅∠P″D′D	(5) Statement 4
(6) ∠PD′D is a right angle	(6) Given

(7) $\angle P''D'D$ is a right angle	(7) Statements 5 and 6
(8) $\angle PD'D$, $P''D'D$ form a linear pair	(8) Statements 6 and 7 and since these are adjacent angles
(9) P, D′, P″ are collinear	(9) Statement 8
(10) P, D, P″ are collinear	(10) Statement 2 and since $\angle PDD'$, $\angle P''DD'$ form a linear pair
(11) Two distinct lines $\overleftrightarrow{PD}$ and $\overleftrightarrow{PD'}$ have two points P and P″ common	(11) Statements 9 and 10.

The result at which we have arrived at in statement 11 contradicts Postulate 2. Hence the assumption that there exist two distinct perpendiculars from P to $\overleftrightarrow{AB}$ has to be abandoned. Hence there exist atmost one perpendicular from P to $\overleftrightarrow{AB}$.

Definition VII (4). (*Bisector of an Angle*). A ray $\overrightarrow{AF}$ is bisector of $\angle CAD$ if F is an interior point of $\angle CAD$ and $\angle CAF \cong \angle FAD$.

Theorem VII (7). Every angle has exactly one bisector. (In a traditional course it is simply assumed that an angle has a bisector.)

Given : An arbitrary angle $\angle CAB$.

To prove that : (1) $\angle CAB$ has atleast one bisector.

(2) $\angle CAB$ has atmost one bisector.

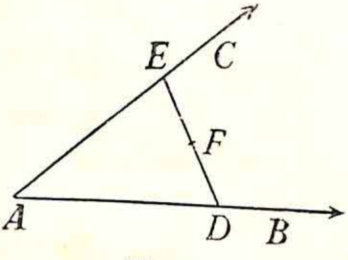


Fig. 93.

Proof.

Statement	*Reason*
(1) There exists a point D on $\overrightarrow{AB}$ such that $m\overline{AD}=r$ where r is any real number	(1) Segment Construction Theorem
(2) There exists a point E on $\overrightarrow{AC}$ such that $m\overline{AE}=r$	(2) Segment Construction Theorem

(3) $\overline{AD}\cong\overline{AE}$	(3) Statements 1 and 2
(4) F is a point on $\overline{ED}$ such that E—F—D and $\overline{EF}\cong\overline{FD}$ (F is mid point of $\overline{ED}$)	(4) Theorem IV (8)
(5) F is an interior point of $\angle$CAD	(5) E—F—D
(6) $\overline{AD}\cong\overline{AE}$ $\overline{AF}\cong\overline{AF}$ $\overline{FD}\cong\overline{EF}$	(6) Statements 3, 4 and every segment is congruent to itself
(7) 1—1 correspondence AFD$\leftrightarrow$AFE is a congruence and $\triangle$AFD$\cong$ $\triangle$AFE	(7) S.S.S. Theorem
(8) $\angle$FAD$\cong\angle$FAE	(8) Corresponding parts of congruent triangles
(9) $\overrightarrow{AF}$ is the angle bisector of $\angle$CAB	(9) Statement 8.

Therefore, there exists atleast one bisector of $\angle$CAB. Now we want to prove that $\angle$CAB has at the most one bisector.

We intend to prove it by contradiction. Assume that it has two bisectors $\overrightarrow{AG}$ and $\overrightarrow{AG'}$ meeting $\overline{ED}$ in F′ and F″ respectively.

Now:

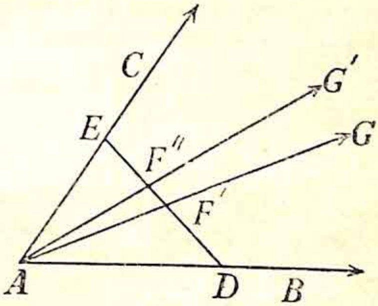


Fig. 94.

Statement	*Reason*
(1) $\overrightarrow{AG}$ is a ray such that $\overrightarrow{AG}$ lies between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and $\angle$F′AD$\cong\angle$EAF′	(1) $\overrightarrow{AG}$ is the angle bisector of $\angle$EAD
(2) $\overrightarrow{AG}$ intersects $\overline{ED}$ at F′ such that E—F′—D	(2) Crossbar Theorem VI (8)
(3) $\overline{AE}\cong\overline{AD}$ AF′$\cong$AF′ and $\angle$F′AE$\cong\angle$F′AD	(3) (*a*) Statement 3 of the first part (*b*) Any segment is congruent to itself (*c*) Statement 1

(4) 1—1 correspondence F'AE↔F'AD is a congruence and $\triangle F'AE \cong \triangle F'AD$	(4) S.A.S. Postulate
(5) $F'E \cong F'D$	(5) Corresponding parts of congruent triangles
(6) F' is mid-point of $\overline{ED}$	(6) Statement 5.

As $\overrightarrow{AG'}$ is another angle bisector of $\angle$CAB, by proceeding on similar lines as above we can prove that F″ is also mid-point of $\overline{ED}$. So we establish that $\overline{ED}$ has two distinct mid-points F′ and F″ but this contradicts the result of the theorem IV (8) where we proved that every segment has exactly one mid-point. Hence the assumption that $\angle$CAB has two angle bisectors $\overrightarrow{AG}$ and $\overrightarrow{AG'}$ has to be abandoned and hence $\angle$CAB has only one angle bisector.

Let Us Review. We have used the term 'congruence' in three different situations.

(1) We used it in connection with line-segments. If we had two line segments $\overline{AB}$ and $\overline{CD}$ such that their lengths or measures

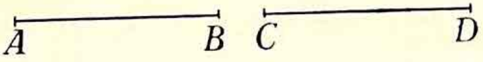


Fig. 95.

were equal, we said that the line segment $\overline{AB}$ is congruent to the line segment $\overline{CD}$. In a course in conventional geometry the term 'equal to' is generally used in such a situation.

(2) If we had two angles P and Q whose measures were equal, we said that the angle P is congruent to angle Q while in a conventional course it will be said that angle P is equal to angle Q. Now the question arises as to why the modern mathematician uses the relationship of 'congruent to' instead of the relation 'equal to'? The difficulty that arises for the mathematician is how to make the difference between $P \cong Q$ as is the case above and P is same as Q (Point to point P is same as Q). The relation involved in 'P is same as Q' is called the logical identity. The reader may be thinking that if 'P is same as Q' then why to call a particular thing, idea, etc.,

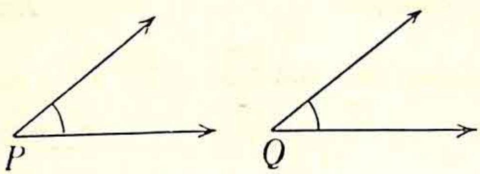


Fig. 96.

by two names P as well as Q? Now the number $(7-3)$ and the number $(2+2)$ describe the same number 4. So we write that $(7-3) = (2+2)$. The relation denoted by the symbol $=$ here is a logical identity. The concept of logical identity or that of $P=Q$ meaning as 'P is same as Q' is very important in advanced modern Maths. and the symbol $(=)$ has been reserved for expressing the relation of 'logical identity'.

The relationship of cogruence is the basic relation in geometry and the special symbol $(\cong)$ has been given to it. If we have an angle $\angle POR$ (as in this figure) and there is a point A on $\overrightarrow{OP}$ and a point B on $\overrightarrow{OR}$, then the same angle has two different names, $\angle POR$ and $\angle AOB$. In this case we write that $\angle POR = \angle AOB$. But if we have two circles with centres A and B and having the same radius and if we denote the first circle by C_1 and the second circle by C_2 then we will write the relation $C_1 \cong C_2$.

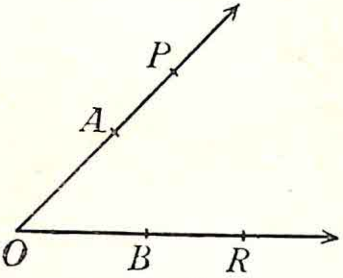


Fig. 97.

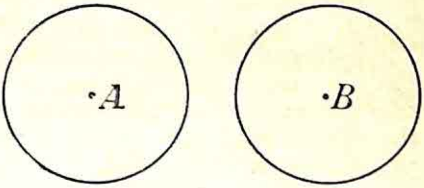


Fig. 98.

(3) As in the case of segments and angles, the congruence relationship between triangles is reflexive as any triangle is congruent to itself. Also it is symmetric as if a $\triangle ABC \cong \triangle DEF$ then $\triangle DEF \cong \triangle ABC$. It is also transitive as if $\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle PQR$ then $\triangle ABC \cong \triangle PQR$. Hence congruence is an equivalence relation in case of triangles. In fact congruence is an equivalence relation in case of any set of geometrical figures.

(4) We have learnt that $1-1$ correspondence $ABC \leftrightarrow DEF$ between the vertices of two triangles is a congruence if six simple congruence relations, three between the corresponding sides and three between the corresponding angles hold. $ABC \leftrightarrow DEF$ implies that the vertex A corresponds to the vertex D, the vertex B corresponds to vertex E and the vertex C corresponds to vertex F. The congruence between corresponding sides and angles will be :

$$\overline{AB} \cong \overline{DE} \quad \text{and} \quad \angle A \cong \angle D$$
$$\overline{AC} \cong \overline{DF} \quad \angle B \cong \angle E$$
$$\overline{BC} \cong \overline{EF} \quad \angle C \cong \angle F.$$

To establish the correspondence between the vertices of two triangles, we need not establish all the six simple congruence relations between the pairs of corresponding sides and angles. Establishing the congruence of certain sets of three of these is enough. These sets are :

(*i*) S.A.S. (*ii*) A.S.A. (*iii*) S.S.S. with these notations having their usual meanings. Later on we will establish two more *viz.*, (*iv*) S.A.A. or A.A.S. (*v*) Hypotenuse-leg in the case of two right triangles.

(5) To write a proof of a theorem involving the use of congruence of the correspondence between the vertices of two triangles, we need to draw a figure denoting each of the vertices by a capital letter and to write (*i*) 'Given' and (*ii*) 'To prove that'. We are not to write any steps of construction. We will be drawing the figure just as an aid to help us in writing down the steps of proof in a logical order. We will not allow the 'proof' to depend on the 'obviousness' of the figure. While writing down the proof, we shall give the reason for every step and this implies indicating the result of a particular theorem that we have already proved or the postulate that we have already stated on which depends the particular step of proof that we are writing.

While making use of the congruence of the correspondence between the vertices of two triangles, we shall set up the correspondence between the vertices of two triangles in such a way that we make use of any one of (*i*) S.A.S. or (*ii*) A.S.A. or (*iii*) S.S.S. depending upon the set of three simple congruence relations that we are in a position to establish.

Problem Set

A

(1) Given that 1—1 correspondence ABC↔QRP between the vertices of two triangles ABC and PQR is a congruence. Then fill in the blanks :

(*i*) A↔ —— (*ii*) B↔ —— (*iii*) C↔ ——

(*iv*) $\angle A \cong$ —— (*v*) $\angle B \cong$ —— (*vi*) $\angle C \cong$ ——

(*vii*) $\overline{AB} \cong$ —— (*viii*) $\overline{AC} \cong$ —— (*ix*) $\overline{BC} \cong$ ——

(2) Given that 1—1 correspondence BAC↔QRP between the vertices of two triangles ABC and PQR is a congruence. List the six pairs of corresponding congruent parts of the triangles.

(3) Given that 1—1 correspondence EDF↔NLM between the vertices of two triangles DEF, LMN is a congruence. Write the six pairs of corresponding congruent parts of these triangles.

(4) List of six pairs of corresponding congruent parts of two triangles, the 1—1 correspondence between whose vertices is a congruence, is given below. Write down this correspondence and also the names of congruent triangles :

$\angle S \cong \angle T$ $\qquad \overline{SU} \cong \overline{TV}$

$\angle U \cong \angle V$ $\qquad \overline{UW} \cong \overline{YU}$

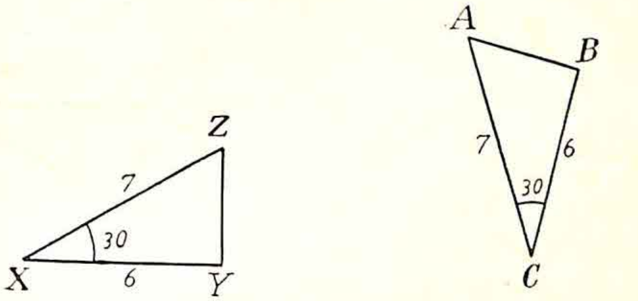
$\angle W \cong \angle Y$ $\qquad \overline{SW} \cong \overline{TV}$

(5) How many 1—1 correspondences can exist between the vertices of two triangles ? How many of these will be congruences if the two triangles are each (*i*) Scalene (*ii*) Isosceles (*iii*) Equilateral in case one of the correspondences happens to be a congruence ?

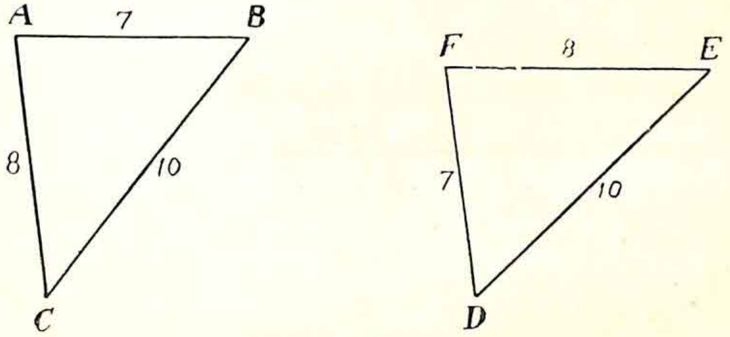
(6) How many 1—1 correspondences can exist between the vertices of a triangle and itself ? How many of these will be congruences if the given triangle is a (*i*) scalene (*ii*) isosceles (*iii*) equilateral ?

B

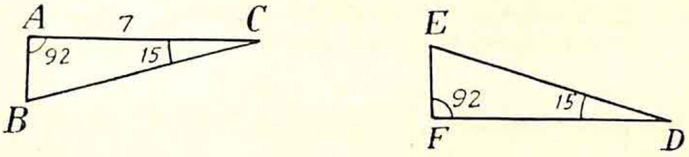
(1) For each pair of triangles given below, write why the triangles are congruent. Also write the 1—1 correspondence between the vertices of the triangles that is a congruence and six congruence relations between the pairs of corresponding sides and angles. (Here the measures of the sides are given in cms. and the measure of angles are given in degrees.)



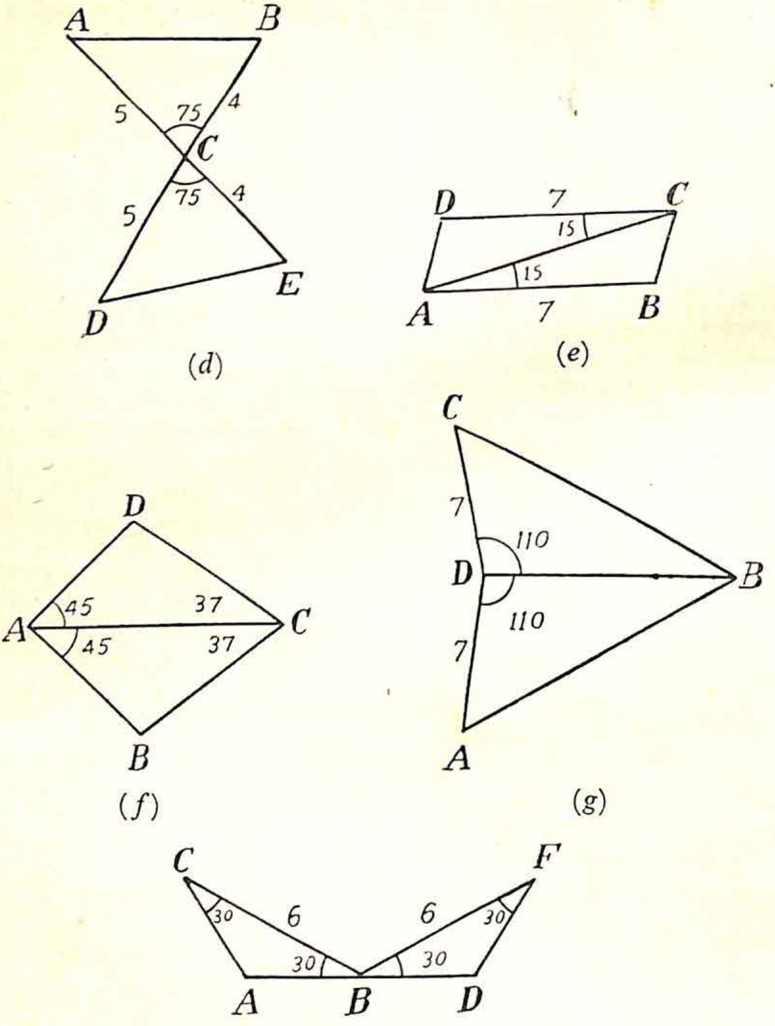
(*a*)



(*b*)



(*c*)



(d) (e) (f) (g) (h)

Fig. 99.

C

(1) Two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at a point E such that A—E—B and C—E—D and $\overline{AE}\cong\overline{EB}$; $\overline{CE}\cong\overline{ED}$. Prove that 1—1 correspondence AEC↔BED is a congruence and $\triangle AEC\cong\triangle BED$.

(2) Given an angle $\angle PMQ$ with a point N on the ray $\overrightarrow{MQ}$ such that M—N—Q and a point O on the ray $\overrightarrow{MP}$ such that

$M-O-P$ and $\overline{MN}\cong\overline{MO}$ and $\overline{MP}\cong\overline{MQ}$. Prove that $1-1$ correspondence $MPN\leftrightarrow MQO$ is a congruence and $\triangle MPN\cong\triangle MQO$.

(3) Four concurrent coplanar rays $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ such that $\overrightarrow{OB}$ lies between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ and $\overrightarrow{OC}$ lies between $\overrightarrow{OB}$ and $\overrightarrow{OD}$. Also $\angle DOC\cong\angle BOA$; $\overline{OD}\cong\overline{OC}$; $\overline{OB}\cong\overline{OA}$. Prove that $1-1$ correspondence $ODB\leftrightarrow OCA$ is a congruence and $\triangle ODB\cong\triangle OCA$.

(4) Given two lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersecting at E such that $A-E-C$ and $D-E-B$. Also $\overline{AD}\cong\overline{BC}$ and $\angle DAB\cong\angle CBA$. Prove that $1-1$ correspondence $DAB\leftrightarrow CBA$ is a congruence and $\triangle DAB\cong\triangle CBA$.

(5) Given a $\triangle ABC$ with a point D on $\overleftrightarrow{BC}$ such that $B-D-C$ and $\overline{BD}\cong\overline{DC}$ and $\angle ADB\cong\angle ADC$ (each of $\angle ADB$, $\angle ADC$ being 90°). Prove that $1-1$ correspondence $ABD\leftrightarrow ACD$ is a congruence and $\triangle ABD\cong\triangle ACD$.

(6) Given a line $\overleftrightarrow{AB}$ dividing a plane π into two half planes H_1 and H_2 and a point C in H_1 and a point D in H_2 such that $\overline{AC}\cong\overline{AD}$ and $\angle CAB\cong\angle BAD$. Prove that $1-1$ correspondence $CAB\leftrightarrow DAB$ is a congruence and $\triangle CAB\cong\triangle DAB$.

(7) Given a $\triangle KEH$ and points F, G on $\overline{EH}$ such that $E-F-G-H$ and $\overline{KF}\cong\overline{KG}$ and $\overline{KH}\cong\overline{KE}$ and $\angle HKG\cong\angle EKF$. Prove that $1-1$ correspondence $KFH\leftrightarrow KGE$ is a congruence and $\triangle KFH\cong\triangle KGE$.

(8) Four concurrent coplanar rays $\overrightarrow{BF}$, $\overrightarrow{BH}$, $\overrightarrow{BR}$ and $\overrightarrow{BA}$ such that $\overrightarrow{BF}$ is opposite to $\overrightarrow{BA}$ and $\overrightarrow{BH}$ is between $\overrightarrow{BF}$ and $\overrightarrow{BR}$ and $\overline{BR}$ is between $\overrightarrow{BH}$ and $\overrightarrow{BA}$. Also $\overline{BF}\cong\overline{BA}$; $\overline{BH}\cong\overline{BR}$ and $\angle HBF\cong\angle RBA$. Prove that $1-1$ correspondence $HBF\leftrightarrow RBA$ is a congruence and $\triangle HBF\cong\triangle RBA$.

(9) Given $\overleftrightarrow{AD}$ divides a plane π into two half planes H_1 and H_2. Two points B, C are on $\overleftrightarrow{AD}$ such that $A-B-C-D$. E, F are two points in H_1 such that $\angle EBD\cong\angle FCA$; $\overline{EB}\cong\overline{FC}$ and $\overline{BD}\cong\overline{CA}$. Prove that $1-1$ congruence $EBD\leftrightarrow FCA$ is a congruence and $\triangle EBD\cong\triangle FCA$.

(10) Every equilateral triangle is equiangular.

(**Definition.** If all the three angles of a triangle are congruent in pairs, then the triangle is known as equiangular.)

Given : $\triangle ABC$ in which $\overline{AB}\cong\overline{BC}$, $\overline{BC}\cong\overline{AC}$ and $\overline{AB}\cong\overline{AC}$.

To prove that : $\angle A\cong\angle B$; $\angle B\cong\angle C$ and $\angle C\cong\angle A$.

Proof. Consider the $1-1$ correspondence $CAB\leftrightarrow CBA$.

Statement	*Reason*
(1) $\overline{CB}\cong\overline{CA}$	(1) Given
(2) $\overline{CA}\cong\overline{CB}$	(2) Given
(3) $\angle C\cong\angle C$	(3) Any angle is congruent to itself
(4) $1-1$ correspondence $CAB\leftrightarrow CBA$ is a congruence and $\triangle CAB\cong\triangle CBA$	(4) S.A.S. Postulate
(5) $\angle A\cong\angle B$	(5) Corresponding parts of congruent triangles

(6) Similarly by considering the correspondence $ABC\leftrightarrow ACB$ we can prove that $\angle B\cong\angle C$ and by considering the corresponding $BCA\leftrightarrow BAC$, we can prove that $\angle C\cong\angle A$.

(11) Given an equilateral triangle ABC and D, E, F are the mid. points of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively. Prove that $\triangle DEF$ is equilateral.

(12) Given a $\triangle ABC$. $\overline{CD}$ the median of $\triangle ABC$ is perpendicular to side $\overline{AB}$. Prove that $\triangle ABC$ is isosceles.

[**Definition.** (*Median*). The segment determined by a vertex of the triangle as one end point and the mid. point of the opposite side as the second end point is called a median of the triangle.]

(13) Prove that the median from the vertex of an isosceles triangle is bisector of the vertical angle.

(14) Prove that the bisector of the angle at the vertex of an isosceles triangle is the perpendicular bisector of the base.

D

(1) Two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in O such that $A-O-B$ and $C-O-D$. Also $\overline{AO}\cong\overline{OB}$ and $\angle OAC\cong\angle OBD$. Prove that $1-1$ correspondence $OAC\leftrightarrow OBD$ is a congruence and $\triangle OAC\cong\triangle BOD$.

(2) Given an $\angle ACB$. A point E on the ray $\overrightarrow{CA}$ such that $C-E-A$ and a point D on the ray $\overrightarrow{CB}$ such that $C-D-B$. Also $\overline{AC}\cong\overline{BC}$ and $\angle DAC\cong\angle EBC$. Prove that $1-1$ correspondence $ACD\leftrightarrow BCE$ is a congruence and $\triangle ACD\cong\triangle BCE$.

(3) Given four concurrent coplanar rays $\overrightarrow{CA}$, $\overrightarrow{CD}$, $\overrightarrow{CE}$, $\overrightarrow{CB}$ such that $\overrightarrow{CD}$ is between $\overrightarrow{CA}$ and $\overrightarrow{CE}$ and $\overrightarrow{CE}$ is between $\overrightarrow{CD}$ and $\overrightarrow{CB}$. Also $\overline{AC} \cong \overline{BC}$; $\angle ACD \cong \angle BCE$ and $\angle CAD \cong \angle CBE$. Prove that $1-1$ correspondence $CAD \leftrightarrow CBE$ is a congruence and $\triangle CAD \cong \triangle CBE$. Further if the points A, D, E and B are collinear, prove that $1-1$ correspondence $CAE \leftrightarrow CBD$ is a congruence and $\triangle CAE \cong \triangle CBD$.

(4) Given a $\triangle ACD$ with $\overline{DB} \perp \overline{AC}$. Also $\angle A \cong \angle C$ and $\overline{AB} \cong \overline{BC}$. Prove that $1-1$ correspondence $ABD \leftrightarrow CBD$ is a congruence and $\triangle ABD \cong \triangle CBD$.

(5) Given an angle $\angle BCA$. Also $BD \perp AC$ such that $C-D-A$. Also E is a point on $\overline{BD}$ such that $B-E-D$ and $\overline{CD} \cong \overline{DE}$. Also $\angle C \cong \angle DEA$. Prove that $1-1$ correspondence $ADE \leftrightarrow BDC$ is a congruence and $\triangle ADE \cong \triangle BDC$.

(6) Given a $\triangle ABC$. D, E are the mid. points of the sides $\overline{AC}$, $\overline{BC}$ respectively. F is a point on $\overleftrightarrow{DE}$ such that $D-E-F$ and $\overline{DE} \cong \overline{EF}$ and $\overline{BF} \cong \overline{AD}$. Write down $1-1$ correspondence that will be a congruence and the triangles that will be congruent.

(7) Given a $\triangle CAB$. G, E are the mid. points of sides $\overline{CA}$ and $\overline{AB}$ respectively. Also F is a point on $\overleftrightarrow{GE}$ such that $G-E-F$, $\overline{GE} \cong \overline{FE}$ and $\angle CGE \cong \angle EFB$. Write down the $1-1$ correspondence that will be a congruence and the triangles that will be congruent to each other. Write down the proof in detail in support of your answer.

(8) Given two lines $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ intersecting at O such that $A-O-B$ and $C--O-D$. Also E is a point on $\overleftrightarrow{AB}$ such that $A-E-O-B$; F is a point on $\overleftrightarrow{CD}$ such that $C-F--O-D$. Also $\angle DEA \cong \angle BFC$ and $\overline{OF} \cong \overline{OE}$. Prove that $\overline{OD} \cong \overline{OB}$.

(9) If the bisector of $\angle A$ of a $\triangle ABC$ is perpendicular to the side $\overline{BC}$, prove that the given triangle is isosceles.

(10) Given a $\triangle ABC$. A point D on the side $\overline{AB}$ and a point E on the side $\overline{AC}$ such that $\angle ADE \cong \angle AED$ and $\angle EDC \cong \angle DEB$. Prove that $\overline{BE} \cong \overline{DC}$.

E

(1) Given an isosceles triangle ABC with $\overline{CA} \cong \overline{CB}$. D is the mid. point of side $\overline{AB}$. Prove that $\overleftrightarrow{CD} \perp \overline{AB}$.

(2) Given a four sided figure ABCD such that $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$. Prove that $\overrightarrow{AC}$ is the angle bisector of $\angle DAB$.

(3) Given a line $\overleftrightarrow{AB}$ dividing a plane π into two half planes H_1 and H_2. Points C and D are in H_1 such that $\overline{AC} \cong \overline{BD}$ and $\overline{BC} \cong \overline{AD}$. Prove that $\angle C \cong \angle D$.

(4) Given a $\triangle ABC$. D is a point on the base $\overline{BC}$ such that $B-D-C$ and $\angle BAD \cong \angle CAD$. E is a point on $\overline{AD}$ such that $A-E-D$ and $\angle BED \cong \angle CED$. Prove that $\overline{AD} \perp \overline{BC}$.

(5) Two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E such that $A-E-B$ and $C-E-D$. $\overrightarrow{EF}$ is a ray between $\overrightarrow{EA}$ and $\overrightarrow{EC}$ such that $\angle CEF \cong \angle AEF$. G is a point on the ray $\overrightarrow{EX}$ which is opposite to the ray $\overrightarrow{EF}$. H is a point on $\overrightarrow{EB}$ $(E-H-B)$ and K is a point on $\overrightarrow{ED}$ $(E-K-D)$ such that $\angle HGE \cong \angle KGE$. Prove that $\overline{GH} \cong \overline{GK}$.

(6) For proving that the base angles of an isosceles triangle are congruent, Euclid started the proof as follows :—

Let ABC be the triangle such that $\overline{AB} \cong \overline{AC}$. Let D be a point on $\overrightarrow{AB}$ such that $A-B-D$ and let E be a point on $\overrightarrow{AC}$ such that $A-C-E$ and such that $\overline{AD} \cong \overline{AE}$. He went on to establish $1-1$ correspondence $ADC \leftrightarrow AEB$ is congruence. Complete this proof.

(7) Prove that all equilateral triangles are

(*a*) equiangular (*b*) isosceles.

F

(1) Prove that the medians bisecting the congruent sides of an isosceles triangle are congruent.

(2) If the diagonals of a quadrilateral bisect each other at right angles, prove that the quadrilateral is a rhombus. (Rhombus is a quadrilateral whose all sides are equal in measure.)

(3) If the diagonals of a quadrilateral are congruent and also bisect each other, prove that it is a rectangle. (Rectangle is a quadrilateral whose opposite sides are congruent and whose all angles are right angles.)

(4) Show that the mid-points of the sides of a square determine another square.

CHAPTER VIII

GEOMETRIC INEQUALITIES AND SOME OTHER PROPERTIES OF THE TRIANGLE

Till now we have been talking about congruence of segments, angles or triangles. In this chapter we propose to take up the segments or the angles that are unequal. If the measures of two segments $\overline{AB}$ and $\overline{CD}$ are unequal, then the measure of one of them will be less than that of the other. Suppose $m\overline{AB} < m\overline{CD}$, then there will exist a point E on $\overline{CD}$ such that C−E−D and $\overline{AB} \cong \overline{CE}$ (segment construction theorem). Similarly if measures of two angles $\angle ABC$ and $\angle DEF$ are unequal then the measure of one of them will be less than the measure of the other. Supposing that $m\angle ABC < m\angle DEF$.

Then there exists a ray $\overrightarrow{EG}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ such that $\angle ABC \cong \angle GEF$. (Why ?)

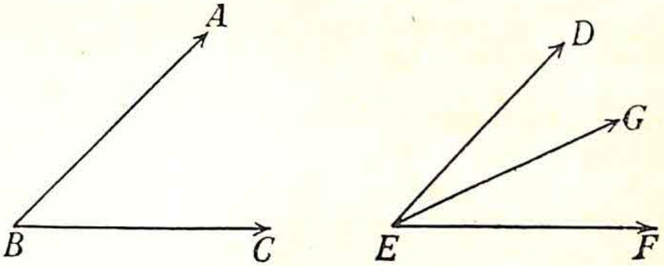


Fig. 100.

Before we explore the properties of a triangle having segments and angles whose measures are unequal, we are to state a definition and prove a theorem or two which we will make use of in proving certain theorems later on.

Definition VIII (1). (*Exterior angle*). Given a triangle ABC. Consider a point D such that B−C−D. Then $\angle ACD$ is called the exterior angle of $\triangle ABC$.

Every triangle has six such exterior angles. These form vertical pairs (vertically congruent angles). At each vertex, the two vertical angles are congruent to each other. $\angle A$ and $\angle B$ are called non-adjacent interior or remote interior angles of the exterior angle with vertex at C. Similarly $\angle B$ and $\angle C$ are the non-adjacent interior angles

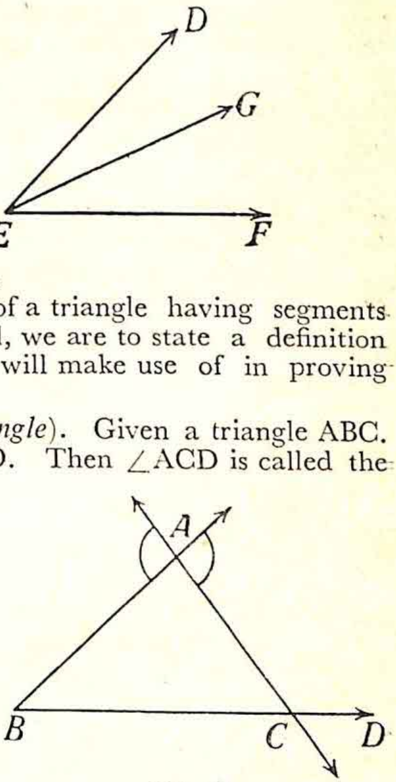


Fig. 101.

of the exterior angle with vertex at A and so on. The interior angle at the vertex is said to be adjacent to the exterior angle at the vertex.

Pasch's Axiom. Supposing there is a $\triangle ABC$ and there is a point D in $\overline{BC}$ such that $B-D-C$. Supposing a ray enters the interior of $\triangle$ at D. Then what happens to the ray ? Euclid is absolutely silent on the matter. Euclid's postulates or Axioms do not help us to know as to what happens. Will it proceed to intersect $\overline{AC}$, then turn and enter through one of the vertices ? Of course in a conventional course of geometry we rely on the figure and assume that either it will pass through the vertex A or it will intersect either one of $\overline{AC}$ or $\overline{AB}$. German mathematician Moritz Pasch (1843—1931) was the first to recognise this lacuna in the Euclid's Postulates. To rectify the defect, he stated the following postulate which is known as Pasch's Axiom.

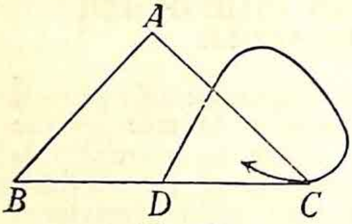


Fig. 102.

Pasch's Axiom. A line lying in the plane of a given triangle and which contains one interior point of one of its sides and which does not pass through any of its vertices, contains an interior point of exactly one other side of the triangle.

Pasch stated it as a postulate but with the help of the postulates that we have already stated, it is possible to prove it as a theorem. The proof is as follows :

Given : A triangle ABC lying in plane π and a line l lying also in plane π such that l contains an interior point D of $\overline{BC}$ but it does not pass through any of the vertices A, B or C.

To prove that : (1) It contains an interior point of either $\overline{AB}$ or $\overline{AC}$.

(2) It contains an interior point of only one of $\overline{AB}$ or $\overline{AC}$.

Proof.

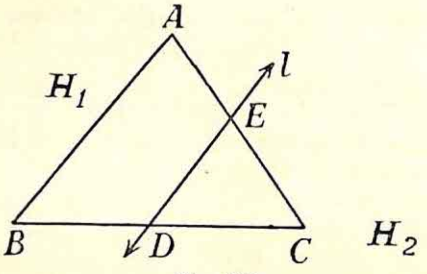


Fig. 103.

Statement	*Reason*
(1) The line l divides the plane π into two half planes H_1 and H_2	(1) Plane Separation Postulate

(2) The line $\overleftrightarrow{BC}$ contains a point D of l	(2) Given
(3) B and C lie in the opposite half planes ; if B lies in H_1 then C lies in H_2	(3) Statement 2 ; Plane Separation Postulate
(4) $A \notin l$	(4) Given : Since l does not pass through any of the vertices
(5) So either A lies in H_1 or H_2	(5) Plane Separation Postulate
(6) If A lies in H_1 then $\overline{AB}$ lies completely in H_1 and hence does not intersect l while $\overline{AC}$ will intersect l as A lies in H_1 and C lies in H_2 Now assume that A lies in H_2	(6) Plane Separation Postulate
(7) If A lies in H_2 then A, B lie in opposite half planes and hence $\overline{AB}$ intersects l but as A, C lie in H_2 so $\overline{AC}$ does not intersect l	(7) Plane Separation Postulate
(8) l intersects one and only one of $\overline{AB}$ or $\overline{AC}$	(8) Statements 6 and 7.

Corollary to Pasch's theorem. A line which passes through one of the vertices of a triangle and contains an interior point of the $\triangle$ also, contains an interior point of the side opposite to the vertex through which it passes.

This corollary helps us to do away with any possibility of a line entering through a vertex, then intersecting one of the sides and then intersecting the opposite side in an exterior point. We can prove this corollary by giving the following argument. $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are concurrent rays and $\overrightarrow{AD}$ is

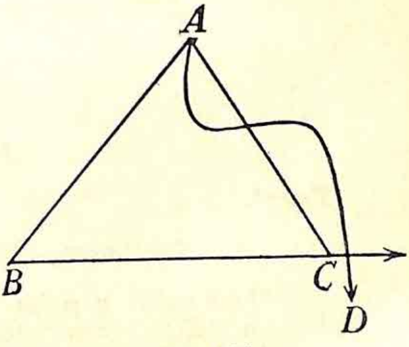


Fig. 104.

between $\overrightarrow{AC}$ and $\overrightarrow{AB}$ as $\overrightarrow{AD}$ contains an interior point of the triangle. Therefore excepting A, all the points of $\overrightarrow{AB}$, $\overrightarrow{AC}$ lie on opposite sides of $\overrightarrow{AD}$. In other words if $\overleftrightarrow{AD}$ divides the plane of the triangle into two half planes H_1 and H_2 and if $\overrightarrow{AB}$ (excepting A) lies in H_1 then $\overrightarrow{AC}$ (excepting A) lies in H_2. It means if B lies in H_1, then C lies in H_2. Therefore $\overline{BC}$ will intersect $\overrightarrow{AD}$. The corrollary to Pasch's theorem is same in substance as Crossbar Theorem VI (8) that we have already proved.

Now we prove some theorems pertaining to geometric inequalities.

Theorem VIII (1). The measure of an exterior angle of a triangle is greater than the measure of either of the remote interior angles.

Given : A $\triangle ABC$. D is a point on $\overleftrightarrow{BC}$ such that $B-C-D$.

To prove that : $m\angle ACD > m\angle B$ and $m\angle ACD > m\angle A$.

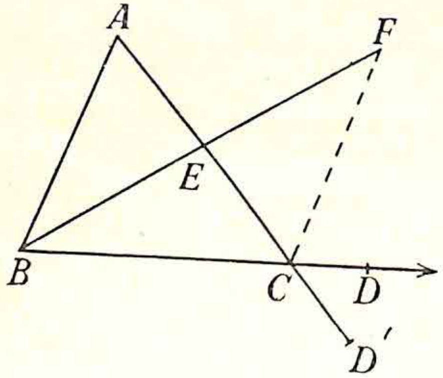


Fig. 105.

Proof.

Statement	*Reason*
(1) There exists a point E on $\overline{AC}$ such that $A-E-C$ and E is mid-point of $\overline{AC}$	(1) Theorem IV (5)

(2) On $\overleftrightarrow{BE}$, there exists a point F such that B—E—F and $\overline{BE} \cong \overline{EF}$	(2) Segment Construction Theorem
(3) F is in the interior of $\angle ACD$	(3) Theorem VI (4)
(4) $m\angle ACD = m\angle ACF + m\angle FCD$	(4) Angle Addition Postulate
(5) $m\angle ACD > m\angle ACF$ or $m\angle ECF$	(5) Statement 4 and Def. of inequality $>$
(6) $\overline{EC} \cong \overline{EA}$, $\overline{EF} \cong \overline{EB}$ $\angle FEC \cong \angle AEB$	(6) Statements 1 and 2 and vertical angles
(7) 1—1 correspondence $FEC \leftrightarrow BEA$ is a congruence and $\triangle FEC \cong \triangle BEA$	(7) S.A.S. Postulate
(8) $\angle ECF \cong \angle EAB$	(8) Corresponding parts of congruent triangles
(9) $m\angle ACD > m\angle EAB$ or $m\angle A$	(9) Statements 5 and 8.

By taking a point D′ on $\overleftrightarrow{AC}$ such that A—C—D and by proceeding on the same lines as above we can prove that $m\angle BCD' > m\angle B$ but $\angle BCD' \cong \angle ACD$ (vertical angles). Hence $m\angle ACD > m\angle B$.

The theorem that we have proved above is known as the weak form of the exterior angle theorem.

Theorem VIII (2). In a right triangle, each of the angles other than the right angle is acute. (This theorem has a significance as so far we have not proved the angle-sum property of a triangle.)

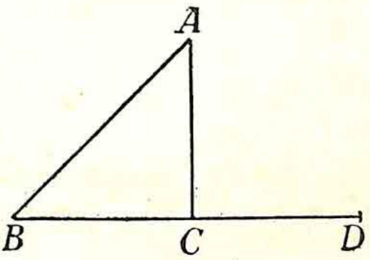


Fig. 106.

Given : A $\triangle ABC$ with $m\angle C = 90$.

To prove that : (*i*) $m\angle A < 90$.

(*ii*) $m\angle B < 90$.

Proof. Consider a point D on $\overleftrightarrow{BC}$ such that B—C—D.

Statement	*Reason*
(1) $\angle ACD$, $\angle ACB$ form a linear pair	(1) $\overrightarrow{CD}$ opposite to $\overrightarrow{CB}$ and $\overrightarrow{CA}$ is any other ray.
(2) $m\angle ACD+m\angle ACB=180$	(2) Statement 1, Theorem VI (1)
(3) $m\angle ACB=90$	(3) Given
(4) $m\angle ACD=90$	(4) Statements 2 and 3
(5) $m\angle ACD>m\angle A$ and $m\angle ACD>m\angle B$	(5) Theorem VIII (1)
(6) $m\angle A<90$ and $m\angle B<90$	(6) Statements 4 and 5.

Theorem VIII (3). If the measures of two sides of a triangle are unequal then the measures of angles opposite these sides are also unequal in the same order in which the measures of sides are unequal.

Given : A $\triangle ABC$ in which $m\ \overline{AB}>m\ \overline{AC}$ or $AB>AC$.
To prove that : $m\angle ACB>m\angle ABC$.

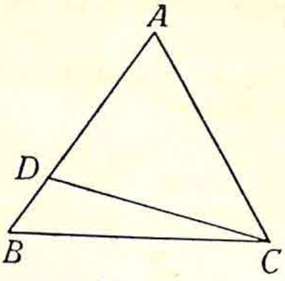


Fig. 10 .

Proof.

Statement	*Reason*
(1) There exists a point D on $\overline{AB}$ such that A—D—B and $\overline{AD}\cong\overline{AC}$	(1) Segment Construction Theorem
(2) D lies on A-side of $\overline{BC}$	(2) Since B is not between A and D ; Statement 1
(3) D lies on B-side of $\overline{AC}$	(3) Since A is not between B and D

(4) D lies in the interior of $\angle ACB$	(4) Statements 2 and 3 and definition of interior of an angle
(5) $\overrightarrow{CD}$ is between $\overrightarrow{CA}$ and $\overrightarrow{CB}$	(5) Statement 4
(6) $m\angle ACB = m\angle ACD + m\angle DCB$	(6) Angle Addition Postulate
(7) $m\angle ACB > m\angle ACD$	(7) Statement 6
(8) $\angle ACD \cong \angle ADC$	(8) $\overline{AC} \cong \overline{AD}$; Statement 1 ; Hence $\triangle ACD$ is isosceles
(9) $m\angle ACB > m\angle ADC$	(9) Statements 7 and 8
(10) $m\angle ADC > m\angle ABC$	(10) Since $\angle ADC$ is an exterior angle for $\triangle BCD$
(11) $m\angle ACB > m\angle ABC$	(11) Statements 9 and 10.

Theorem VIII (4). [*Converse to Theorem VIII (3)*]. If the measures of two angles of a triangle are unequal, the measures of the sides opposite these angles are unequal in the same order in which the measures of angles are unequal.

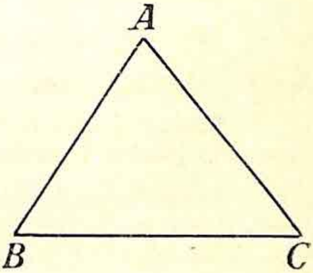


Fig. 108.

Given : A $\triangle ABC$ in which

$$m\angle ABC > m\angle ACB.$$

To prove that : $m\overline{AC} > m\overline{AB}$ or $AC > AB$.

Proof. There are three possible relations between the measures of sides $\overline{AB}$ and $\overline{AC}$.

(1) $m\overline{AC} = m\overline{AB}$ or $AC = AB$.

(2) $m\overline{AC} < m\overline{AB}$ or $AC < AB$.

(3) $m\overline{AC} > m\overline{AB}$ or $AC > AB$.

First Case. If $m\overline{AC} = m\overline{AB}$ then $\angle ABC \cong \angle ACB$ (why ?) which contradicts the hypothesis that $m\angle ABC > m\angle ACB$. Hence the assumption that $m\ \overline{AC} = m\ \overline{AB}$ needs to be abandoned.

Second Case. If $m\overline{AC} < m\overline{AB}$ then $m\angle ABC < m < ACB$ (why ?) which contradicts the hypothesis that $mABC > m\angle ACB$. Hence the assumption that $m\overline{AC} < m\overline{AB}$ needs to be abandoned.

Hence the third possibility that $m\overline{A\ } > m\overline{AB}$ is accepted.

Theorem VIII (5). The shortest segment joining a point to a line (point not lying on the given line) is the perpendicular segment.

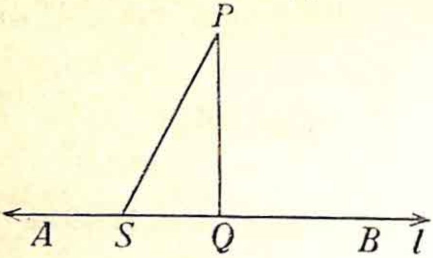


Fig. 109.

Given : A line $\overleftrightarrow{AB}$ and a point P $\notin$ $\overleftrightarrow{AB}$. A point Q $\in \overleftrightarrow{AB}$ such that $\overline{PQ} \perp \overleftrightarrow{AB}$ and any other point S $\in$ $\overleftrightarrow{AB}$.

To prove that : $m\overline{PQ} < m\overline{PS}$.

Proof.

Statement	*Reason*
(1) $\triangle PQS$ is right angled with $m\angle Q = 90$	(1) Given
(2) $m\angle P < 90$ and $m\angle S < 90$	(2) Theorem VIII (2)
(3) $m\angle Q > m\angle S$	(3) Statements 1 and 2
(4) $m\overline{PS} > m\overline{PQ}$	(4) Theorem VIII (4)
(5) $m\overline{PQ} < m\overline{PS}$	(5) Statement 4.

Since S is an arbitrary point with $S \neq Q$, $\overline{PQ}$ is the shortest segment joining P to the given line.

Sometimes while writing proofs of certain theorems, we are to consider a perpendicular from a vertex to opposite side. We must know whether the foot of the perpendicular is in the interior of the opposite side or in its exterior. In a conventional course, we are to depend on a figure. If the figure is of the form as left figure of Fig. 110 the foot of the perpendicular from P on the side $\overline{QR}$ will be an interior point of $\overline{QR}$. If the figure is of the form as right figure of Fig. 110 the foot of the perpendicular from the vertex P to the side

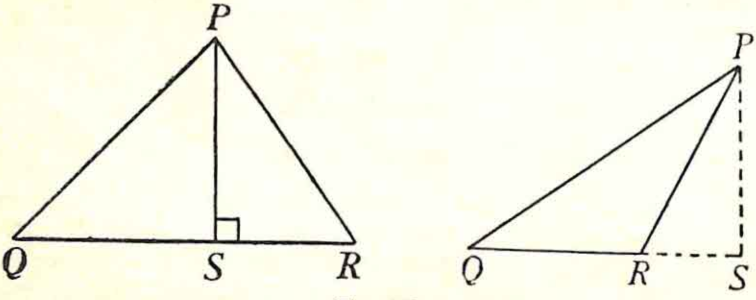


Fig. 110.

$\overline{QR}$ will be an exterior point of $\overline{QR}$. In the present course, we will be exactly stating as to when the foot of such a perpendicular will be an interior point of the opposite side and when it will be an exterior point. For this purpose, we prove the following theorem.

Theorem VIII (6). If one side of a triangle is atleast as long as either of the other two sides, then the foot of the perpendicular from the opposite vertex to the line determined by that side will be an interior point of that side.

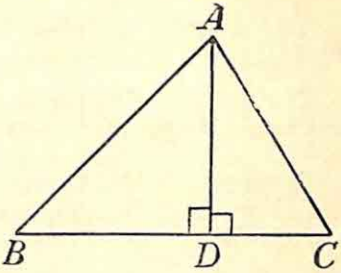


Fig. 111.

Given : $\triangle$ABC in which (*i*) BC$\geqslant$AC and (*ii*) BC$\geqslant$AB and D is the foot of the perpendicular from A to $\overline{BC}$.

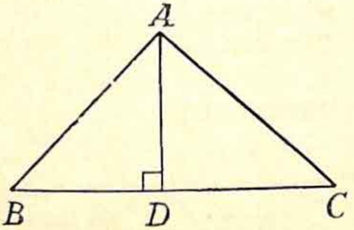
To prove that : D is an interior point of $\overline{\ \ C}$ or B$-$D$-$C.

Proof.

Statement	*Reason*
(1) In a right $\triangle$ADB, $m\angle BAD < m\angle ADB$	(1) Theorem VIII (2)
(2) BD$<$AB	(2) Statement 1 Theorem VIII (3)
(3) CD$<$AC	(3) Reasoning as for statements 1 and 2
(4) AB$\leqslant$BC and AC$\leqslant$BC	(4) Given
(5) BD$<$BC	(5) Statements 2 and 4
(6) CD$<$BC	(6) Statements 3 and 4
(7) B$-$D$-$C	(7) Statements 5 and 6
(8) D is interior point of $\overline{BC}$	(8) Statement 7.

So we have established that the foot of the perpendicular from a vertex of a triangle to the opposite side will be an interior point of this side if this is atleast as long as either of the other two sides. Now we make use of this result to prove the following theorem.

Theorem VIII (7). (*The Triangle inequality*). The sum of lengths of any two sides of a triangle is greater than the third side.



Fig, 112.

Given : $\triangle$ABC.

To prove that :

(1) AB+BC$>$AC

(2) AB+AC$>$BC

(3) BC+AC$>$AB.

Proof. Let $\overline{BC}$ be a side which is atleast as long as $\overline{AB}$ or $\overline{AC}$.

Therefore (*i*) $BC \geqslant AB$ and (*ii*) $BC \geqslant AC$.

As $BC \geqslant AB$, therefore $BC+AC \geqslant AB+AC$. Also $AC > 0$.

Hence $BC+AC > AB$ and thus we prove result (3).

We also know that $BC \geqslant AC$, therefore $BC+AB \geqslant AC+AB$.

As $AB > 0$, therefore $BC+AB > AC$ and thus we prove result (1).

Now to prove the result (2), let D be the foot of the perpendicular from A to $\overline{BC}$. Also $B-D-C$ [Theorem VIII (6)].

Moreover $AB > BD$ (why ?) and $AC > DC$ (why ?) and hence on adding these inequalities we get that $AB+AC > BD+DC$ or $AB+AC > BC$.

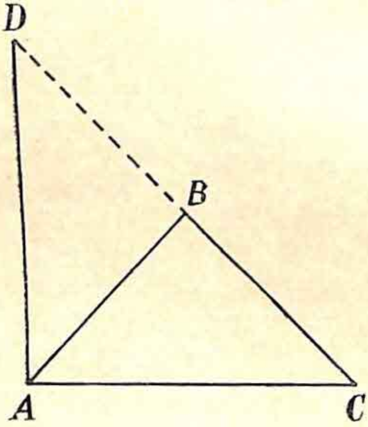


Fig. 113.

Alternative Proof.

Statement	*Reason*
(1) D is a point on $\overrightarrow{CB}$ such that $C-B-D$ and $\overline{BD} \cong \overline{BA}$	(1) Segment Construction Theorem
(2) $\triangle ABD$ is isosceles	(2) Statement 1
(3) $\angle D \cong \angle BAD$	(3) Statement 2
(4) B is on D-side of $\overrightarrow{AC}$ and C-side of $\overrightarrow{AD}$	(4) Statement 1
(5) B is in the interior of $\angle CAD$	(5) Statement 4

(6) $m\angle DAC = m\angle DAB + m\angle BAC$	(6) Angle Addition Postulate
(7) $m\angle DAC > m\angle DAB$	(7) Statement 6
(8) $m\angle DAC > m\angle D$	(8) Statements 3 and 7
(9) $CD > AC$	(9) Theorem VIII (4)
(10) $CB + BD > AC$	(10) Statement 9 and as $C-B-D$
(11) $CB + AB > AC$	(11) Statements 1 and 10.

Similarly we can prove the other two relations.

We have already proved A.S.A. theorem of congruency of two triangles but this is a weak form of the theorem on congruence involving two pairs of corresponding congruent angles and a pair of corresponding congruent sides. In its general form not only A.S.A. but A.A.S. or S.A.A. will be a congruence with the notation having its usual meaning. After having proved the weak form of the exterior angle theorem, we are now ready to prove S.A.A theorem.

Theorem VIII (8). (*S.A.A. Theorem*). If there exists a $1-1$ correspondence between two triangles (or between a $\triangle$ and itself) in which two angles and a side of one triangle are congruent to corresponding parts of the other triangle, the correspondence is a congruence and the triangles are congruent.

Given : Two triangles, $\triangle ABC$ and $\triangle DEF$ in which $A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$. Also $\overline{AB} \cong \overline{DE}$, $\angle B \cong \angle E$ and $\angle C \cong \angle F$.

To prove that : $1-1$ correspondence $ABC \leftrightarrow DEF$ is a congruence and $\triangle ABC \cong \triangle DEF$.

Proof. The measures of the side $\overline{BC}$ of $\triangle ABC$ and the side $\overline{EF}$ of $\triangle DEF$ bear either one of the following relations :

(1) $BC > EF$ (2) $BC < EF$ (3) $\overline{BC} \cong \overline{FF}$ or $BC = EF$.

First case. If $BC > EF$ then :

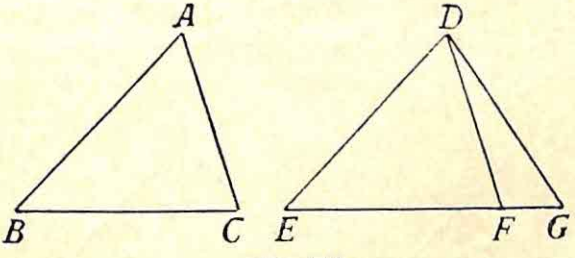


Fig. 114.

Statement	*Reason*
(1) There is a point G on $\overrightarrow{EF}$ such that $E-F-G$ and $BC = EG$ (or $\overline{BC} \cong \overline{EG}$)	(1) Segment Construction Theorem

(2) (*a*) $\overline{AB}\cong\overline{DE}$ (*b*) $\overline{BC}\cong\overline{EG}$ and (*c*) $\angle B\cong\angle E$	(2) (*a*) Given (*b*) Statement 1 (*c*) Given
(3) 1—1 correspondence $ABC\leftrightarrow DEG$ is a congruence and $\triangle ABC\cong\triangle DEG$	(3) S.A.S. Postulate
(4) $\angle C\cong\angle DGE$	(4) Corresponding parts of congruent triangles
(5) $m\angle F>m\angle DGE$	(5) Weak form of exterior angle theorem
(6) $m\angle F>m\angle C$	(6) Statements 4 and 5.

The result which we have arrived at in statement 6 contradicts the hypothesis that $\angle C\cong\angle F$. Hence the assumption that $BC>EF$ needs to be abandoned.

Second Case. If $BC<EF$: By proceeding on the same lines as in the first case we can prove that this assumption also leads to a contradiction. Hence this also needs to be abandoned.

Third Case. If $\overline{BC}\cong\overline{EF}$, then 1—1 correspondence $ABC\leftrightarrow DEF$ is a congruence and $\triangle ABC\cong\triangle DEF$ (S.A.S. Postulate). Hence we establish the congruence relation between $\triangle ABC$ and $\triangle DEF$ (S.A.A. form or A.A.S. form).

Theorem VIII (9). (*Hypotenuse Leg theorem*). If there exists a 1—1 correspondence between two right triangles (or between a triangle and itself) with the right angle of one having a correspondence with the right angle of the other and if hypotenuse and one leg of one triangle are congruent to the corresponding parts of the other triangle, then the correspondence is a congruence and the given triangles are congruent to each other.

Given : $\triangle ABC$ and $\triangle A'B'C'$ such that (*i*) $m\angle C=m\angle C'=90$. (*ii*) $\overline{AB}\cong\overline{A'B'}$. (*iii*) $\overline{BC}\cong\overline{B'C'}$. (*iv*) $ABC\leftrightarrow A'B'C'$.

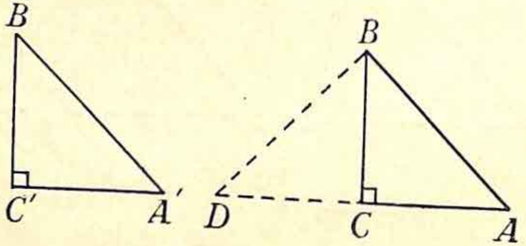


Fig. 115.

To prove that : 1—1 correspondence $ABC\leftrightarrow A'B'C'$ is a congruence and $\triangle ABC\cong\triangle A'B'C'$.

Proof.

Statement	*Reason*
(1) On $\overrightarrow{AC}$, there exists a point D such that $\overline{CD} \cong \overline{C'A'}$ such that D—C—A	(1) Segment Construction Theorem
(2) $\overrightarrow{CD}$ is opposite to $\overrightarrow{CA}$ and $\overrightarrow{CB}$ is any other ray	(2) Statement 1
(3) $m\angle BCA + m\angle BCD = 180$	(3) Statement 2
(4) $m\angle BCA = 90$	(4) Given
(5) $m\angle BCD = 90$	(5) Statements 3 and 4
(6) $m\angle B'C'A' = 90$	(6) Given
(7) $\angle B'C'A' \cong \angle BCD$	(7) Statements 5 and 6
(8) $\overline{B'C'} \cong \overline{BC}$	(8) Given
(9) $\overline{C'A'} \cong \overline{CD}$	(9) Statement 1
(10) 1—1 correspondence B'C'A' $\leftrightarrow$BCD is a congruence and $\triangle B'C'A' \cong \triangle BCD$	(10) S.A.S. Postulate
(11) $\overline{B'A'} \cong \overline{BD}$	(11) Corresponding parts of congruent triangles
(12) $\overline{B'A'} \cong \overline{BA}$	(12) Given
(13) $\angle ADB \cong \angle BAC$	(13) As $\overline{BD} \cong \overline{BA}$ Statements 11 and 12
(14) $\overline{BC} \cong \overline{BC}$	(14) Any segment is congruent to itself
(15) $\angle BCA \cong \angle BCD$	(15) Statements 4 and 5
(16) 1—1 correspondence BCA$\leftrightarrow$ BCD is a congruence and $\triangle BCA \cong \triangle BCD$	(16) S.A.A. Theorem
(17) $\angle CBA \cong \angle CBD$	(17) Why ?
(18) $\angle CBD \cong \angle C'B'A'$	(18) Statement 10
(19) $\angle CBA \cong \angle C'B'A'$	(19) Why ?
(20) $\triangle ABC \cong \triangle A'B'C'$	(20) S.A.S. Postulate.

Let us Review : In this chapter we have discussed that :—

(1) If the measures of two segments are unequal and if there are two segments $\overline{AB}$ and $\overline{CD}$ such that $m\overline{AB} < m\ \overline{CD}$ then there exists a point $E \in \overline{CD}$ such that C—E—D and $\overline{AB} \cong \overline{CE}$.

(2) If the measures of two angles are unequal and if there are two angles $\angle ABC$ and $\angle DEF$ such that $m\angle ABC < m\angle DEF$ then

there exists a ray $\overrightarrow{EG}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ such that $\angle ABC \cong \angle GED$.

(3) We have discussed Pasch's Postulate (or Pasch's Theorem) which states that a line lying in the plane of a $\triangle$ and which does not pass through any of the vertices of the triangle and which if contains an interior point of one of the sides of the triangle, then it contains an interior point of exactly one other side of the triangle. But if the line passes through one of the vertices of the triangle and also contains an interior point of it, it contains an interior point of the side opposite the vertex through which it passes.

(4) We have also proved that weak form of exterior angle theorem which states that the measure of an exterior angle of a triangle is greater than the measure of either of the remote interior angles.

(5) We made use of the weak form of exterior angle theorem to prove that (*i*) In a right triangle, each of its angles other than the right angle is acute and (*b*) If two sides of a triangle are not congruent then the angles opposite these sides are not congruent and the larger angle is opposite the larger side and conversely and (*c*) The triangle inequality theorem and (*d*) S.A.A. theorem on congruency.

(6) We have made use of S.A.A. theorem to prove the hypotenuse-leg theorem on congruency of two right triangles.

(7) We have also proved that the shortest segment joining a point to a line is the perpendicular segment.

(8) We have also established that the foot of the perpendicular drawn from a vertex to its opposise side, lies in the interior of that side if the opposite side is atleast as long as either of the two sides of the given triangle.

Problem Set

A

(1) Given A line $\overleftrightarrow{AE}$ dividing a plane π into two half planes H_1 and H_2. Also given a point B on $\overleftrightarrow{AE}$ such that A$-$B$-$E. In H_1, there are two points C and D such that $\overrightarrow{AD}$ and $\overrightarrow{BC}$ intersect at O such that A$-$O$-$D and B$-$O$-$C and $\overline{AO} \cong \overline{OD}$ and $\overline{BO} \cong \overline{OC}$.

Prove that (*a*) $\angle DBO \cong \angle ACO$ (*b*) $m\angle CBE > m\angle DBO$ (*c*) $m\angle CBE > m\angle ACO$.

(2) Prove that :—

(*a*) The exterior angles of an equilateral triangle are equal in measure.

(*b*) The exterior angles of an equiangular triangle are equal in measure.

(3) Prove that if one exterior angle of a triangle equals in measure to its adjacent interior angle, then the triangle is a right triangle.

(4) Prove that the sum of the measures of any two angles of a triangle is less than 180.

Given : $\triangle ABC$.

To prove that :

(1) $m\angle B + m\angle C < 180$

(2) $m\angle C + m\angle A < 180$

(3) $m\angle A + m\angle B < 180$.

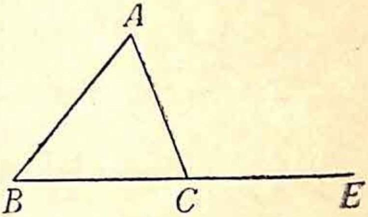


Fig. 116.

Proof.

Statement	*Reason*
(1) $m\angle ACE + m\angle C = 180$	(1) $\angle ACE$ and $\angle C$ form a linear pair
(2) $m\angle ACE > m\angle A$	(2) Exterior Angle Theorem
(3) $m\angle ACE > m\angle B$	(3) Exterior Angle Theorem
(4) $m\angle A + m\angle C < 180$	(4) Statements 1 and 2
(5) $m\angle B + m\angle C < 180$	(5) Statements 1 and 3.
(6) Similarly we can prove that $m\angle A + m\angle B < 180$	

(5) With the help of the statement proved in problem 4, prove that the base angles of an isosceles triangle are acute. Is it possible to prove that the base angles of any triangle are acute? If not, why not?

(6) Prove that the sum of lengths of diagonals of any quadrilateral (four sided figure) is less than the sum of lengths of its sides.

(7) If A, B, C are any three non-collinear points then prove that (*i*) $AB + BC > AC$ (*ii*) $AB + AC > BC$ (*iii*) $AC + BC > AB$. Also prove that $AB + BC = AC$ iff B is on $\overline{AC}$ and $A-B-C$.

(8) ABC is a $\triangle$ with $\overline{AB} \cong \overline{AC}$. D is a point on $\overline{BC}$ such that $B-C-D$. Prove that (*i*) $AD > AB$ (*ii*) $AD > AC$.

(9) $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have a point O in common such that $A-O-B$ and $C-O-D$. E is a point in the plane of these lines. Prove that $EA + EB + EC + ED > OA + OB + OC + OD$.

(10) Prove that if two sides of one triangle are congruent respectively to two sides of a second triangle then the measure of third

side of first triangle will be larger than the measure of the third side of the second triangle iff the measure of the angle determined by the two sides of the first triangle is greater than the measure of the angle determined by the two sides of the second triangle.

Solution. First we will prove the 'if' part.

Given : A $\triangle ABC$ and a $\triangle DEF$ such that $\overline{AB}\cong\overline{DE}$, $\overline{AC}\cong\overline{DF}$ and $m\angle A>m\angle D$.

To prove that : $m\overline{BC}>m\overline{EF}$.

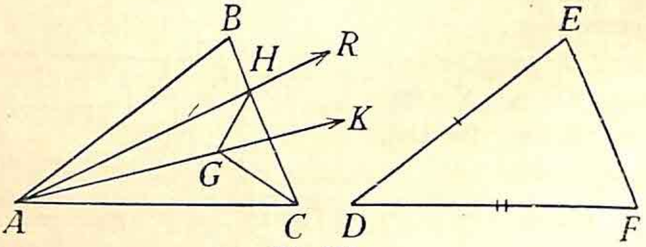


Fig. 117.

Proof.

Statement	*Reason*
(1) There exists a ray $\overrightarrow{AK}$ such that $\overrightarrow{AK}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$ and such that $\angle KAC\cong\angle D$	(1) Angle Construction Postulate
(2) There exists a point G on $\overrightarrow{AK}$ such that $\overline{AG}\cong\overline{DE}$	(2) Segment Construction Theorem
(3) $\overline{AC}\cong\overline{DF}$	(3) Given
(4) 1—1 correspondence $GAC\leftrightarrow EDF$ is a congruence and $\triangle GAC\cong\triangle DEF$	(4) S.A.S. Postulate
(5) $\overline{CG}\cong\overline{EF}$	(5) Corresponding parts of congruent triangles
(6) $\overrightarrow{AR}$ is a ray between $\overrightarrow{AK}$ and $\overrightarrow{AB}$ having a point H in common with BC and such that $\angle BAH\cong\angle HAG$ and C—H—B	(6) Angle Construction Postulate ;Pasch's Theorem

(7) $\overline{AG}\cong\overline{AB}$	(7) $\overline{AG}\cong\overline{DE}$ and $\overline{DE}\cong\overline{AB}$
(8) $\overline{AH}\cong\overline{AH}$	(8) Any segment is congruent to itself
(9) $1-1$ correspondence $ABH\leftrightarrow AGH$ is a congruence and $\triangle ABH\cong\triangle AGH$	(9) S.A.S. Postulate
(10) $\overline{BH}\cong\overline{HG}$	(10) Corresponding parts of congruent triangles
(11) $m\overline{CG}<m\overline{CH}+m\overline{HG}$	(11) Triangle Inequality Theorem
(12) $m\overline{CG}<m\overline{CH}+m\overline{BH}$	(12) Statement 10
(13) $m\overline{CG}<m\overline{CB}$	(13) Statement 6
(14) $m\overline{EF}<m\overline{CB}$	(14) Statements 5 and 13.

Now we prove 'only if part' (Converse to the first part).

Given : $\triangle ABC$ and $\triangle DEF$ in which $\overline{AB}\cong\overline{DE}$, $\overline{AC}\cong\overline{DF}$ and $m\overline{BC}>m\overline{EF}$.

To prove that : $m\angle A>m\angle D$.

Proof. There are three possibilities regarding the relationship between the measures of $\angle A$ and $\angle D$.

(*i*) $m\angle A=m\angle D$. (*ii*) $m\angle A<m\angle D$. (*iii*) $m\angle A>m\angle D$.

First Case. If $m\angle A=m\angle D$ then $1-1$ correspondence $ABC\leftrightarrow DEF$ is a congruence and hence $\triangle ABC\cong\triangle DEF$ (S.A.S. Postulate) and hence $m\overline{BC}=m\overline{EF}$ which contradicts the hypothesis that $m\overline{BC}>m\overline{EF}$. Hence the assumption that $m\angle A=m\angle D$ has to be abandoned.

Second Case. If $m\angle A<m\angle D$ then $m\overline{BC}<m\overline{EF}$ (If part) and this also contradicts the hypothesis that $m\overline{BC}>m\overline{EF}$. Hence this assumption has also to be abandoned.

Third Case. $m\angle A>m\angle D$ is accepted.

(11) State separately the 'if' part (Theorem) and 'only if' part (the converse) of the statement given in problem 10 and prove them.

(12) Given a $\triangle ABC$ and a point D in $\overline{AB}$ such that $A-D-B$. Prove that if $\overline{CA}\cong\overline{CB}$ and $m\angle ACD<m\angle BCD$ then $m\overline{AD}<m\overline{DB}$.

(13) Given a $\triangle ABC$ with $m\overline{AB}>m\overline{AC}$ and D a point in $\overline{BC}$ such that $B-D-C$ and $\overline{BD}\cong\overline{DC}$. Prove that $m\angle ADB>m\angle ADC$.

(14) Given a $\triangle ACB$ such that $m\overline{AB}>m\overline{AC}$. D is a point in $\overline{AB}$ such that $A-D-B$ and E is a point on $\overline{AC}$ such that $A-E-C$ and $\overline{EC}\cong\overline{DB}$. Prove that $m\overline{EB}>m\overline{CD}$.

(15) Given a $\triangle ABC$ and a point P in the interior of $\triangle ABC$. Prove that $m\overline{PA}+m\overline{PB}+m\overline{PC}>\frac{1}{2}(m\overline{AB}+m\overline{AC}+m\overline{BC})$.

(16) Given a $\triangle ABC$ such that $m\overline{AB}>m\overline{AC}$ and $m\overline{AB}>m\overline{BC}$. Prove that $m\overline{AB}>\frac{1}{2}(m\overline{AC}+m\overline{BC})$.

(17) Prove that the sum of the measures of three angles is less than 270.

Proof. If $\triangle ABC$ is the given triangle, then we know that

$m\angle A+m\angle B<180$ (Problem 4)

and $m\angle B+m\angle C<180$

and $m\angle C+m\angle A<180$

On adding, we get

$$2(m\angle A+m\angle B+m\angle C)<540$$

or $(m\angle A+m\angle B+m\angle C)<270$.

(18) Given a $\triangle ABC$ and a point P in its interior. Prove that

$$m\overline{PA}+m\overline{PB}<m\overline{CA}+m\overline{CB}.$$

(19) Prove that sum of the measures of altitudes of a triangle is less than the perimeter of the triangle (Fig. 118)

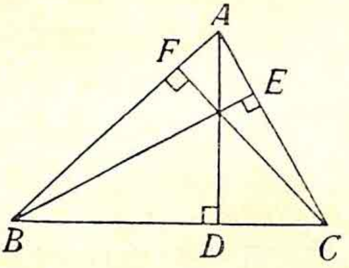


Fig. 118.

Proof.

Statement	*Reason*
(1) $m\overline{AD}<m\overline{AB}$	(1) In $\triangle ADB$ $m\angle B<m\angle ADB$.
(2) $m\overline{BE}<m\overline{BC}$	
(3) $m\overline{CF}<m\overline{CA}$	

On adding $(m\overline{AD}+m\overline{BE}+m\overline{CF})<(m\overline{AB}+m\overline{BC}+m\overline{CA})$.

(20) Given a $\triangle ABC$ and D a point in $\overline{BC}$ such that $B-D-C$. Prove that $m\overline{AD}<m\overline{AB}$ or $m\overline{AD}<m\overline{AC}$.

(21) Given a $\triangle ABC$ such that $\overline{AB}\cong\overline{AC}$. D is a point in $\overline{BC}$ such that $B-D-C$. Prove that $m\overline{AD}<m\overline{AB}$ and $m\overline{AD}<m\overline{AC}$.

(22) Prove that if $\angle B$ in a $\triangle ABC$ is obtuse then the foot of perpendicular from A to $\overline{BC}$ is not a point of the side $\overline{BC}$ of the $\triangle ABC$.

(23) If $\triangle ABC$ and $\triangle DEF$ are such that $m\overline{AB}=m\overline{AC}=m\overline{DE}=m\overline{DF}$ and $m\overline{BC}<m\overline{EF}$, prove that $m\angle BAC<m\angle EDF$. State and prove its converse also.

(24) Given a $\triangle ABC$. Prove that angle bisectors of $\angle B$ and $\angle C$ intersect in a point in the interior of the $\triangle$.

(25) Show that semi-perimeter of a triangle is greater than any of its sides.

(26) Prove that the perimeter of a quadrilateral is greater than the sum of the measures of its two diagonals but the semi-perimeter of the quadrilateral is less than the sum of the measures of its two diagonals.

(27) Prove that the perimeter of a triangle is greater than the sum of the measures of its medians.

(28) Given a $\triangle ABC$. D is mid-point of $\overline{BC}$.

Prove that $m\overline{AB} + m\overline{AC} > 2(m\overline{AD})$.

(29) Assuming that medians of a triangle are concurrent, prove that sum of the measures of medians of a triangle is greater than the semi-perimeter of a triangle.

(30) In a $\triangle ABC$, D is the mid-point of $\overline{BC}$. If $m\overline{AD} > m\overline{BD}$, prove that $\angle A$ is acute.

(31) If two sides of a triangle are unequal in measure then the angle determined by those sides is divided unequally into two parts by the median to the third side such that the smaller part is adjacent to the greater side.

(32) Show that the difference of measures of any two sides of a triangle is less than the measure of the third side.

To prove that : (1) $m\overline{AB} - m\overline{AC} < m\overline{BC}$ if $m\overline{AB} > m\overline{AC}$.

(2) $m\overline{AC} - m\overline{AB} < m\overline{BC}$ if $mAC > mAB$ and four more similar relations.

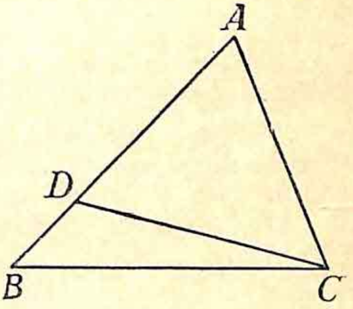


Fig. 119.

Proof. Assume $m\overline{AB} > m\overline{AC}$.

Statement	*Reason*
(1) There is a point D such that $A - D - B$ and $AD \cong \overline{AC}$.	(1) Segment Construction Theorem
(2) $m\angle ADC > m\angle DCB$	(2) Why ?
(3) $m\angle CDB > m\angle ACD$	(3) Why ?
(4) $m\angle ACD = m\angle ADC$	(4) Why ?
(5) $m\angle CDB > m\angle ADC$	(5) Statements 3 and 4
(6) $m\angle CDB > mDCB$	(6) Statements 2 and 5
(7) $m\overline{BC} > m\overline{BD}$	(7) Why ?

(8) $m\overline{BC} > m\overline{AB} - \overline{AD}$	(8) $m\overline{AB} = m\overline{AD} + m\overline{BD}$ as $A-D-B$
(9) $m\overline{AB} - m\overline{AD} < m\overline{BC}$	(9) Statement 8.
(10) $m\overline{AB} - m\overline{AC} < m\overline{BC}$	(10) Statement 1.

(33) In a $\triangle ABC$, $m\overline{AB} > m\overline{AC}$. $\overrightarrow{AX}$ is the angle bisector of $\angle A$ and has a point D in common with $\overline{BC}$. Prove that $m\overline{BD} > m\overline{CD}$.

(34) Prove that if the altitudes of a triangle are equal in measure then the triangle is equilateral.

(35) Prove that if two altitudes of a triangle are equal in measure then the triangle is isosceles.

(36) $\overrightarrow{AX}$, the angle bisector of $\angle A$ of $\triangle ABC$ has a point D in common with the side $\overline{BC}$ such that $B-D-C$ and $\overline{BD} \cong \overline{DC}$. Prove that triangle is isosceles.

(37) $\overline{AD}$ is the perpendicular segment from A to the side $\overline{BC}$ of a $\triangle ABC$ in which $\overline{AB} \cong \overline{AC}$. Prove that $A-D-B$ and $\overline{AD} \cong \overline{DB}$.

(38) In a $\triangle ABC$, $m\angle C = 90$ and $\overline{AC} \cong \overline{BC}$. $\overrightarrow{AY}$ the angle bisector of $\angle A$ meets $\overline{BC}$ in D. Prove that $m\overline{AC} + m\overline{CD} = m\overline{AB}$.

(39) Given two triangles : $\triangle ABC$ and $\triangle DEF$. $\overline{AL}$ is the perpendicular segment from A to $\overline{BC}$ and $\overline{DM}$ is the perpendicular segment from D to $\overline{EF}$ such that $\overline{AL} \cong \overline{DM}$. If $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$ prove the following :

(*a*) $1-1$ correspondence $ABL \leftrightarrow DEM$ is a congruence and $\triangle ABL \cong \triangle DEM$.

(*b*) $1-1$ correspondence $ALC \leftrightarrow DMF$ is a congruence and $\triangle ALC \cong \triangle DMF$.

(*c*) $1-1$ correspondence $ABC \leftrightarrow DEF$ is a congruence and $\triangle ABC \cong \triangle DEF$.

(40) Prove that if the measure of one of the sides opposite an acute angle of a right triangle is half the measure of the hypotenuse then the measure of the acute angle opposite this side is half the measure of the other acute angle.

CHAPTER IX

PARALLEL LINES IN A PLANE

So far we have been talking about the lines that intersect. However in a problem contained in the chapter on 'Incidence Postulates and Theorems' we introduced a postulate which we called Parallel Postulate and this postulate assumed the existence of a line co-planar with a given line but which would not intersect the given line. In this chapter, we intend to talk about the existence of such lines that do not intersect. When we think about the lines that do not intersect, two concepts come to our mind which are that of (1) Parallel lines (2) Skew lines.

Parallel Lines : Parallel lines are sometimes described as lines 'that do not meet however far these are produced or extended' or 'parallel lines are the lines that are equidistant at every point' or 'parallel lines have same direction'. These definitions/descriptions of parallel lines are defective as it is hardly possible to explain the concept of direction in exact mathematical terms (especially in this space age). If at all we may have to try to define the concept of 'direction' then we will have to depend on the concept of parallelism. So we define 'parallelism' in terms of 'direction' and then define 'direction' in terms of parallelism which clearly means that we have not been able to define anything. This type of definition is called a 'round argument' and 'round argument' leads us nowhere. Similarly, it is not possible to define the concept of equidistance. As stated earlier our definitions have to be in terms of those terms which we take as undefined or in those terms which we have already defined. Hence it is convenient for us to base the definition of parallel lines on the idea of 'do not meet' or 'do not have any point in common' or the idea of 'non-intersection'.

Definition IX (1). (*Parallel lines*). Two lines that lie in the same plane (or are co-planar) and which do not intersect are called parallel lines. In such a case each line will be said to be parallel to the other. If l_1 and l_2 are the lines that are parallel then we denote it by writing '$l_1 \parallel l_2$'.

In other words parallel lines must satisfy two conditions namely :

(1) Lines must lie in the same plane, (2) These must not have any point in common.

Lines that do not intersect but also do not lie in the same plane will not be parallel. Such lines are called skew lines.

Definition IX (2). (*Skew lines*). Lines that have no point in common and are not co-planar are called 'skew lines'.

Definition IX (3). (*Parallel segments*). Two segments are said to be parallel if the lines that contain them are parallel. Thus $\overline{AB}$ is parallel to $\overline{CD}$ if $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

It may be clearly understood that two line segments could be coplanar, may not have any point in common and still these may not be parallel. Hence we had to define parallel segments in terms of the lines that contain them. Reference here may be made to the phrase 'however far they are extended or produced'.

Definition IX (4). (*Parallel rays*). Two rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be parallel if (*i*) $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel and (*ii*) B, D be on the same side of $\overleftrightarrow{AC}$.

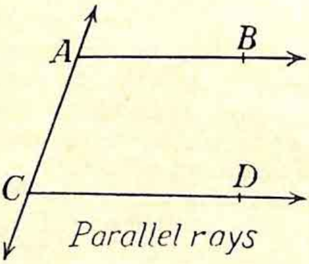


Fig. 120.

It may be understood that the rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ will not be parallel if B and D lie on opposite side of $\overleftrightarrow{AC}$. If $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are rays such that B and D are on the opposite sides of $\overleftrightarrow{AC}$ then such rays are said to be anti-parallel.

Definition IX (5). (*Anti-parallel rays*). Two rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be anti-parallel iff :

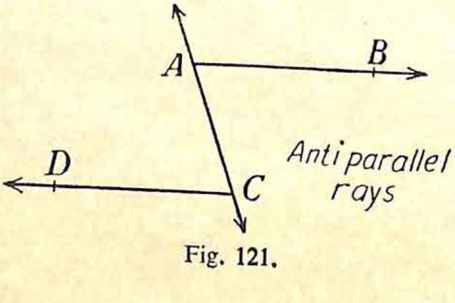


Fig. 121.

(1) The lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel ; and

(2) B and D lie on the opposite side of $\overleftrightarrow{AC}$.

On similar lines we can build up the definition of parallelism between (*i*) a segment and a line (*ii*) a segment and a ray and (*iii*) a ray and a line etc. For instance a segment $\overline{AB}$ and the line $\overleftrightarrow{CD}$ will be parallel if $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel.

One very important notion that helps very much in the study of properties of parallel lines is the notion of 'Transversal' of two lines.

Definition IX (6). (*Transversal*). A transversal of two coplanar lines is a line which intersects the union of two lines in two different points.

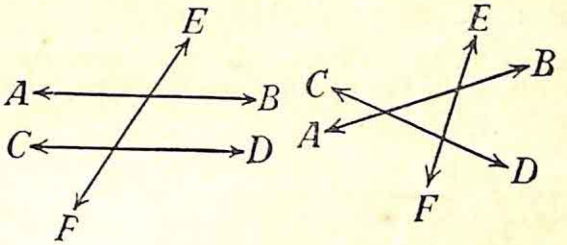


Fig. 122.

Physical models of a transversal.

It may be understood that a line passing through the intersection of two lines will not be a transversal according to the definition of transversal that we have given.

After having defined parallel lines and the transversal we are ready to prove the following theorems.

Theorem IX (1). If p, q are any two coplanar lines and t is any transversal such that $p \perp t$ and $q \perp t$ then $p \parallel q$.

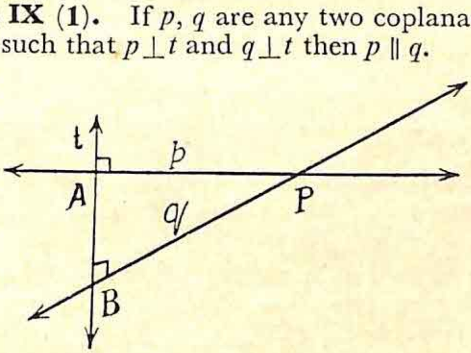


Fig. 123.

Given : Two lines p, q and a transversal t such that $p \perp t$ and $q \perp t$.

To prove that : $p \parallel q$.

Proof. We prove it indirectly by contradiction.

Assume that $p \nparallel q$ (p is not parallel to q). Then :

Statement	*Reason*
(1) p and q have a point in common ; Let P be that point	(1) If p and q are not parallel then these intersect
(2) $\overline{PA} \perp t$	(2) $p \perp t$ (Given)
(3) $\overline{PB} \perp t$	(3) $q \perp t$ (Given).

Statements 2 and 3 lead us to conclude that from a point P ($P \notin t$), there are two perpendicular segments $\overline{PA}$ and $\overline{PB}$ to the same line t. This contradicts the result of the theorem VII (6) which states that there is one and only one line that passes through a given point and is perpendicular to a given line. Hence the assumption that $p \nparallel q$ has to be abandoned. Hence $p \parallel q$.

Theorem IX (2). Given a line and a point not in it. Then there exists atleast one line which passes through the given point and is parallel to the given line.

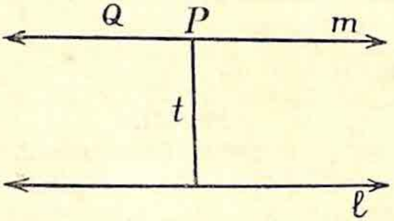


Fig. 124.

Given : A line l lying in a plane π and a point P, $P \notin l$ but $P \in \pi$.

To prove that : (1) There is a line m containing P and $\parallel l$.

Proof.

Statement	*Reason*
(1) There exists a unique line t through P such that $t \perp l$	(1) Theorem VII (6)
(2) There exists a unique ray $\overrightarrow{PQ}$ in π such that $\overrightarrow{PQ} \perp t$ (Angle between $\overrightarrow{PQ}$ and t is 90)	(2) Angle Construction Postulate

(3) $\overrightarrow{PQ} \perp t$ and $l \perp t$	(3) Statements 1 and 2
(4) $\overrightarrow{PQ} \parallel l$	(4) Statement 3 ; Theorem IX (1)
(5) Through P, there exists a line parallel to l	(5) Statement 4.

After having got some insight into concept of parallel lines, we would like to make some definitions in order to make matters easier for us to explore further the properties of parallel lines. In the theorems that we proved above, we dealt with two coplanar lines and a transversal. If we look at the physical model of two coplanar lines and a transversal, we find that various pairs of angles are formed. If, P_1, P_2 are any two points in p ; Q_1, Q_2 are any two points in q ; T_1 is the point in common with p and t ; T_2 is the point in common with q and t and A, B are any two points on t such that (*i*) $P_1—T_1—P_2$ (*ii*) $Q_1—T_2—Q_2$ and (*iii*) $A—T_1—T_2—B$ then :

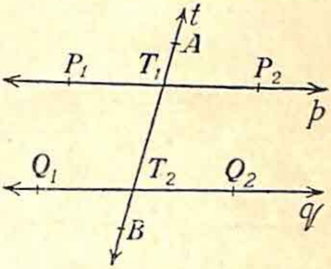


Fig. 125.

(1) $\angle P_2T_1T_2$, $\angle Q_1T_2T_1$ and $\angle P_1T_1T_2$, $\angle Q_2T_2T_1$ are two pairs of angles which have a special name—alternate interior angles.

(2) $\angle AT_1P_2$, $\angle T_1T_2Q_2$; $\angle AT_1P_1$, $\angle T_1T_2Q$; $\angle P_2T_1T_2$; $\angle Q_2T_2B$; $\angle P_1T_1T_2$, $\angle Q_1T_2B$ are the four pairs of corresponding angles.

(3) $\angle P_2T_1T_2$, $\angle Q_2T_2T_1$ and $\angle P_1T_1T_2$, $\angle Q_1T_2T_1$ are the two pairs of consecutive interior angles.

The various pairs of angles have been given these special names as when these pairs exhibit certain properties then the lines will be parallel and these names will help us in making statements in a concise language. We know that the linear pairs are supplementary. With the help of this we can prove that if one pair of alternate angles are congruent then the other pair will also be congruent. Similarly if one pair of corresponding angles are congruent then the remaining three pairs will be congruent. Also if one pair of consecutive interior angles are supplementary then the other pair will also be supplementary. These, of course, are very good problems to be proved but at the moment our concern is to establish conditions for the existence of parallel lines. Do the parallel lines as defined by us exist in our infinite space ? It is a very controversial question. But we shall be limiting ourselves to the finite space around us in our immediate neighbourhood. So we state and prove the following theorems.

Theorem IX (3). If a pair of alternate interior angles formed by two coplanar lines and a transversal are congruent then the given lines are parallel.

Given : Two distinct lines $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ and a transversal $\overleftrightarrow{EF}$ having a point P in common with $\overleftrightarrow{AB}$ and a point Q in common with $\overleftrightarrow{CD}$. Also $\angle BPQ \cong \angle CQP$ and $\angle APQ \cong \angle DQP$.

To prove that : $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

Proof. There are two possibilities in which $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ can behave :

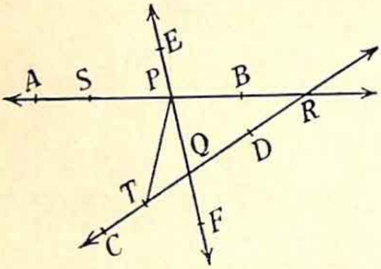


Fig. 126.

(1) $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have a point R in common.

(1) $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ do not have any point in common.

Proof. (1) When $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have a point R in common.

Statement	*Reason*
(1) There exists a point T on the ray opposite to $\overrightarrow{QR}$ such that $\overline{TQ} \cong \overline{RP}$. Also there exists a point S on the ray opposite to $\overrightarrow{PR}$	(1) Segment Construction Theorem
(2) $\angle TQP \cong \angle RPQ$	(2) Given
(3) $\overline{QP} \cong \overline{PQ}$	(3) Any segment is congruent to itself
(4) 1−1 correspondence TQP↔RPQ is a congruence and $\triangle TQP \cong \triangle RPQ$	(4) S.A.S. Postulate
(5) $\angle TPQ \cong \angle PQR$	(5) Corresponding parts of congruent triangles
(6) $\angle SPQ \cong \angle PQR$	(6) Given
(7) $\angle TPQ \cong \angle SPQ$	(7) Statements 5 and 6.

(3) S, T lie on the same side of $\overleftrightarrow{PQ}$	(8) S lies on ray opposite to $\overrightarrow{PR}$ and T lies on the ray opposite to $\overrightarrow{QR}$ Plane Separation Postulate
(9) $\overrightarrow{PT}=\overrightarrow{PS}$ (Ray $\overrightarrow{PT}$ is same as ray $\overrightarrow{PS}$. In conventional terminology such a situation will be described as $\overrightarrow{PT}$ and $\overrightarrow{PS}$ coincide with each other but this description is not very exact.)	(9) Statements 7 and 8
(10) P, T, S are collinear	(10) Statement 9
(11) T lies on $\overleftrightarrow{AB}$	(11) Statement 10 as $\overline{PS}$ is contained in $\overleftrightarrow{AB}$
(12) R lies on $\overleftrightarrow{AB}$	(12) $R \in \overleftrightarrow{AB}$
(13) T, R lie on $\overleftrightarrow{CD}$	(13) Statement 1
(14) T, R are different points	(14) T lies on the ray opposite to $\overrightarrow{QR}$
(15) $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have two points T, R in common	(15) Statements 11, 12, 13, 14
(16) $\overleftrightarrow{AB}=\overleftrightarrow{CD}$	(16) Statement 15 ; Postulate 2.

But $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two distinct lines. Hence $\overleftrightarrow{AB} \neq \overleftrightarrow{CD}$. Hence the assumption that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have a point R in common has to be abandoned.

Case II. The second possibility that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ do not have any points in common is accepted.

Thus we have established that if a pair of alternate interior angles formed by two coplanar lines and a transversal are congruent then the given lines are parallel. We will make use of this result in proving the following theorems.

Theorem IX (4). If a pair of corresponding angles formed by two coplanar lines and a transversal are congruent then the given lines are parallel.

Given : $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ two distinct coplanar lines and a transversal $\overleftrightarrow{EF}$ having a point P in common with $\overleftrightarrow{AB}$ and a point Q in common with $\overleftrightarrow{CD}$ such that (*i*) $\angle EPB \cong \angle PQD$ (*ii*) $\angle EPA \cong \angle PQC$ (*iii*) $\angle BPQ \cong \angle DQF$ (*iv*) $\angle APQ \cong \angle CQF$. (In fact it would have been enough to have mentioned the congruency of one pair of corresponding angles as other relations would have followed from it by the use of theorems about linear pairs and vertical angles.)

To prove that : $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

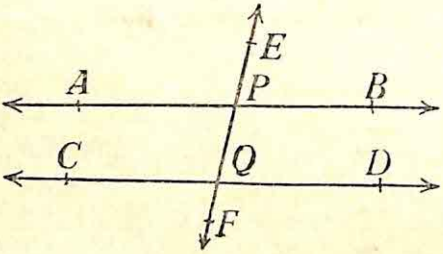


Fig. 127.

Proof.

Statement	*Reason*
(1) $\angle EPB \cong \angle PQD$	(1) Given
(2) $\angle EPB \cong \angle APQ$	(2) Vertical angles
(3) $\angle PQD \cong \angle APQ$	(3) Statements 1 and 2
(4) Pair of alternate angles formed by two coplanar lines and a transversal are congruent	(4) Statement 3
(5) $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	(5) Theorem IX (3).

Theorem IX (5). Two coplanar lines which are perpendicular to the same line are parallel.

Proof.

Its proof immediately follows from Theorem IX (4) since measure of each of the eight angles that are formed in such a situation is 90. Hence a pair of corresponding angles are congruent.

Theorem IX (6). If a pair of consecutive interior angles formed by two coplanar lines and a transversal are supplementary then the given lines are parallel.

Given : $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ two distinct coplanar lines and a transversal $\overleftrightarrow{EF}$ having a point P in common with $\overleftrightarrow{AB}$ and a point Q in common with $\overleftrightarrow{CD}$ such that :

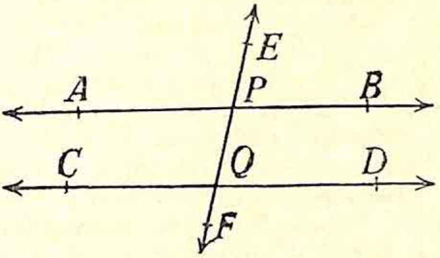


Fig. 128.

(1) $m\angle BPQ + m\angle PQD = 180$
(2) $m\angle APQ + m\angle PQC = 180$.

To prove that : $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

Proof.

Statement	*Reason*
(1) $m\angle BPQ + m\angle PQD = 180$	(1) Given
(2) $\angle BPQ$, $\angle APQ$ form a linear pair	(2) $\overrightarrow{PA}$ is opposite to $\overrightarrow{PB}$ and $\overrightarrow{PQ}$ is any other ray
(3) $m\angle BPQ + m\angle APQ = 180$	(3) Statement 2
(4) $m\angle PQD = m\angle APQ$ or $\angle PQD \cong \angle APQ$	(4) Statements 1 and 3
(5) Pair of alternate angles formed by two coplanar lines and a transversal are congruent	(5) Statement 4
(6) $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	(6) Theorem IX (3).

Euclid's Parallel Postulate. In theorem IX (2), we proved that a given line and a point not in it, there exists at least one line which passess through the given point and parallel to the given line. So we have found the existence of *at least* one line through a point P ($P \notin \overleftrightarrow{AB}$) which is parallel to $\overleftrightarrow{AB}$ in the structure that we are building up. But are there more than one line through P which are parallel to $\overleftrightarrow{AB}$? The use of the word 'atleast' in the theorem cited here may be noted. Can we use the word 'atmost' in the statement of theorem IX (2) and prove it? Alternately can we prove the converse of the theorem IX (3)? Can we prove that if two lines are parallel then the alternate interior angles determined by these lines and any transversal are necessarily congruent? On the basis of the postulates that we have already stated and the theorems that we have already proved, it is not possible to do so. We need another Postulate for this purpose. The postulate that can help us in this situation is the Euclid's Parallel Postulate which has been the basis of controversy between mathematicians since long. This postulate of Euclid was considered to be a law of nature for thousands of years. During the nineteenth century mathematicians like Lobachevski and Bolyai developed geometrical systems by denying that there is one and only one line through P parallel to $\overleftrightarrow{AB}$. They postulated the—existence of atleast two parallel lines through P parallel to any given line $\overleftrightarrow{AB}$. Rieman developed another geometrical system by postulating that no two lines in the same plane are ever parallel. In other words, Rieman's system absolutely denies the existence of parallel lines. But in the system that we are developing or in the Euclidean system the consistency of Euclid's parallel postulate with other postulates and the theorems that we have already developed has been established. Hence we state the following version of Euclid's Parallel Postulate.

Postulate 17. (*The Parallel Postulate*). Given a line and a point not on that line, there is atmost one line through the given point parallel to the given line.

By stating this postulate, we simply assume the uniqueness of parallel lines. In other words we assume that there exists one and only one line through a point P ($P \notin \overleftrightarrow{AB}$) parallel to a given line $\overleftrightarrow{AB}$.

It may however be stated that very interesting and useful non-Euclidean geometries have been developed by postulating the existence of atleast two parallels to a given line through a given point not in that line. Such geometries have been found to have very useful applications in Physics.

Having stated the parallel postulate, we apply it in proving the converse to each of the theorems IX (3), IX (4) and IX (6).

Theorem IX (7). [*Converse to theorem IX (3)*]. The alternate interior angles determined by two parallel lines and any transversal are congruent.

Given : $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and a transversal '*t*' having a point P in common with $\overleftrightarrow{AB}$ and a point Q in common with $\overleftrightarrow{CD}$.

To prove that : (*i*) $\angle BPQ \cong \angle PQC$ (*ii*) $\angle APQ \cong \angle PQD$.

Proof. Now there are three possibilities of the relationship between the measures of $\angle BPQ$ and $\angle PQC$:

(1) $m\angle BPQ > m\angle PQC$
(2) $m\angle BPQ < m\angle PQC$ { Trichotomy Law }
(3) $m\angle BPQ = m\angle PQC$

Case I. If $m\angle BPQ > m\angle PQC$ then :

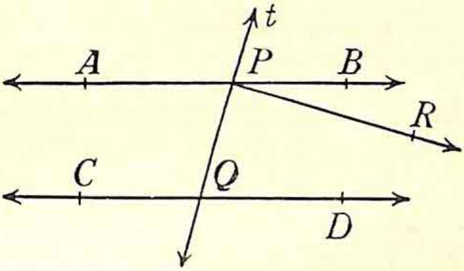


Fig. 129.

Statement	*Reason*
(1) There exists a unique ray $\overrightarrow{PR}$ in the half plane determined by $\overrightarrow{PQ}$ and D such that $\overrightarrow{PR}$ is between $\overrightarrow{PB}$ and $\overrightarrow{PQ}$ and $\angle RPQ \cong \angle PQC$	(1) Angle Construction Postulate and Angle Addition Postulate
(2) $\overleftrightarrow{PR} \parallel \overleftrightarrow{CD}$	(2) Statement 1, Theorem IX (3)
(3) $\overleftrightarrow{PR}$ is different from $\overleftrightarrow{AB}$	(3) Statement 1
(4) $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $P \in \overleftrightarrow{AB}$	(4) Given
(5) Two distinct lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{PR}$ containing a point $P(P \notin \overleftrightarrow{CD})$ are parallel to $\overleftrightarrow{CD}$	(5) Statements 2, 3, 4

The result at which we arrive in statement 5 contradicts the Parallel Postulate which states that there is atmost one line through $P(P \notin \overleftrightarrow{CD})$ parallel to $\overleftrightarrow{CD}$. Hence the assumption that $m\angle BPQ > m\angle PQC$ has to be abandoned.

Case II. If $m\angle BPQ < m\angle PQC$.

By proceeding on the same lines as for case 1, we can prove that this assumption also leads us to a contradiction of Parallel Postulate. Hence this assumption also has to be abandoned.

Case III. If $m\angle BPQ = m\angle PQC$.

As this possibility only remains we accept this.

Hence we prove that the alternate interior angles determined by two parallel lines and any transversal are congruent.

Theorem IX (8). [*Converse to theorem IX (4).*] Corresponding angles determined by two parallel lines and any transversal are congruent.

Given : $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and a transversal 't' having a point P in common with $\overleftrightarrow{AB}$ and a point Q in common with $\overleftrightarrow{CD}$. Also points $T, S \in \overleftrightarrow{PQ}$ such that $T-P-Q-S$.

To prove that : (*i*) $\angle TPA \cong \angle PQC$

(*ii*) $\angle TPB \cong \angle PQD$

(*iii*) $\angle BPQ \cong \angle DQS$

(*iv*) $\angle APQ \cong \angle CQS$

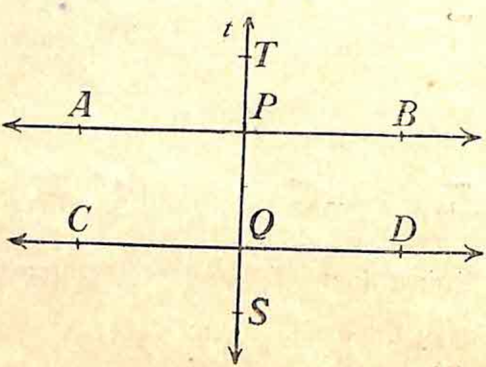


Fig. 130.

Proof.

Statement	*Reason*
(1) $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	(1) Given
(2) $\angle BPQ \cong \angle PQC$	(2) Interior alternate angles [Theorem IX (7)]
(3) $\angle BPQ \cong \angle TPA$	(3) Vertical angles
(4) $\angle TPA \cong \angle PQC$	(4) Statements 2 and 3.

To prove $\angle TPB \cong \angle PQD$, we would take the relation $\angle APQ \cong \angle PQD$ as these are alternate interior angles and as $\angle APQ \cong \angle TPB$ being vertical angles, we arrive at the result that $\angle TPB \cong \angle PQD$.

On similar lines the remaining two results can be proved. The reader may prove these as problems.

Theorem IX (9). [*Converse to theorem IX* (*6*).] Consecutive interior angles determined by two parallel lines and any transversal are supplementary.

Given : $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and a transversal '*t*' having a point P in common with $\overleftrightarrow{AB}$ and a point Q in common with $\overleftrightarrow{CD}$.

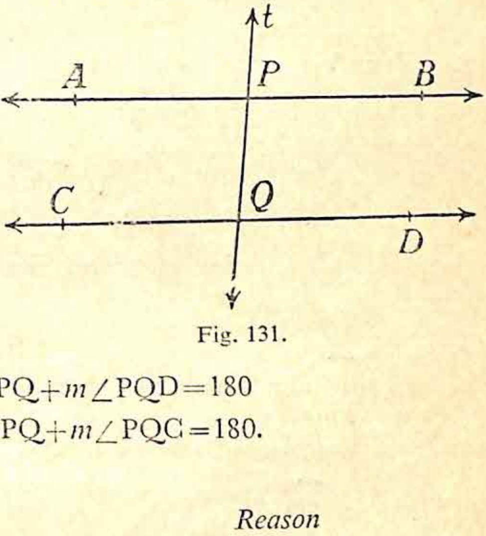


Fig. 131.

To prove that : (1) $m\angle BPQ + m\angle PQD = 180$

(2) $m\angle APQ + m\angle PQC = 180$.

Proof.

Statement	*Reason*
(1) $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	(1) Given
(2) $\angle BPQ \cong \angle PQC$	(2) Alternate interior angles
(3) $m\angle PQC + m\angle PQD = 180$	(3) $\angle PQC$, $\angle PQD$ form a linear pair
(4) $m\angle BPQ + m\angle PQD = 180$	(4) Statements 2 and 3.

Similarly we can prove that $m\angle APQ + m\angle PQC = 180$.

Angle Sum Property of a Triangle. In the problem set on the chapter on geometric inequalities, we proved that the sum of measures of any two angles of any triangle is less than 180. So far, we were not in a position to prove the theorem on the sum of measures of three angles of a triangle. After having stated the parallel postulate and provedtheorem IX (7) we are in a position to prove that the sum of measures of angles of a triangle is 180.

Theorem IX (10). (*The Angle-sum theorem*). The sum of the measures of three angles of any triangle is 180.

Given : $\triangle ABC$.

To prove that : $m\angle A + m\angle B + m\angle C = 180$.

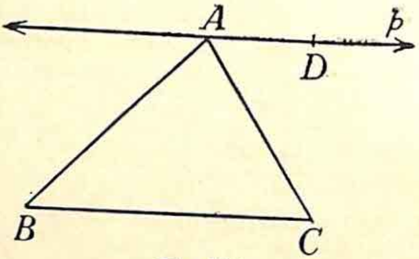


Fig. 132.

Proof.

Statement	*Reason*
(1) Through A, there exists a unique line $p \parallel \overleftrightarrow{BC}$	(1) Parallel Postulate
(2) D is a point on p on the C-side of $\overleftrightarrow{AB}$	(2) Point Plotting theorem
(3) $\overleftrightarrow{BC}$ and p have no points in common	(3) $\overleftrightarrow{BC} \parallel p$
(4) B, C are on the same side of $\overleftrightarrow{AD}$	(4) Statement 3 ; Plane Separation Postulate
(5) $\overrightarrow{AC}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AD}$ or C is in the interior of $\angle DAB$	(5) Statements 2 and 4 as C is on D-side of $\overleftrightarrow{AB}$ and B-side of $\overleftrightarrow{AD}$.
(6) $m\angle DAB = m\angle DAC + m\angle CAB$	(6) Statement 5 ; Angle Addition Postulate
(7) $\angle DAC \cong \angle ACB$	(7) Alternate interior angles

(8) $m\angle DAB = m\angle ACB + m\angle CAB$	(8) Statements 6 and 7
(9) $m\angle DAB + m\angle ABC = 180$	(9) Theorem IX (9)
(10) $m\angle ACB + m\angle CAB + m\angle ABC = 180$	(10) Statements 8 and 9
(11) $m\angle C + m\angle A + m\angle B = 180$	(11) Statement 10
(12) $m\angle A + m\angle B + m\angle C = 180$	(12) Statement 11.

This theorem is most important theorem in Euclidean geometry. In fact we can take this theorem as a Postulate and prove the Parallel Postulate as a theorem. We can say that the parallel postulate and the angle sum theorem (There exists a triangle for which the sum of the measures of the angles is 180) are logically equivalent. It may however be noted that this angle-sum theorem holds only in the system which we are developing (Euclidean geometry). In the non-Euclidean geometry (Plane Hyperbolic geometry) which postulates the existence of atleast two parallel lines through a point to a given line, the sum of the measures of angles of a triangle will be less than 180. In the Plane Hyperbolic geometry, alternate interior angles formed by two parallel lines and a transversal will not be congruent. In another non-Euclidean geometry (Elliptic geometry), the sum of the measures of angles of a triangle will be greater than 180.

We have already proved that the measure of an exterier angle of a triangle is greater than the measure of either of the remote interior angles [Theorem VIII (1)]. We called it the weak form of exterior angle theorem. After having proved the angle-sum theorem, we are in a position to prove the stronger form of the exterior angle theorem whose statement is as follows :—

Theorem IX (11). The measure of an exterior angle of a triangle is equal to the sum of the measures of two non-adjacent interior angles.

Given : A $\triangle ABC$ and a point D in $\overleftrightarrow{BC}$ such that B–C–D.

To prove that : $m\angle ACD = m\angle A + m\angle B$

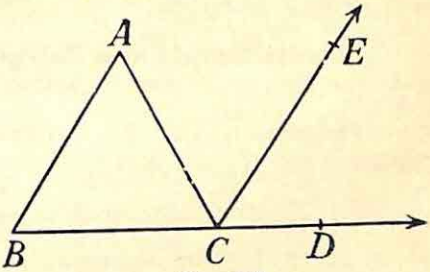


Fig. 133.

Proof.

Statement	*Reason*
(1) Through C, there exists a unique line $\overleftrightarrow{CE}$ parallel to $\overleftrightarrow{AB}$	(1) Parallel Postulate

(2) E is a point on $\overleftrightarrow{CE}$ such that E is on A-side of $\overleftrightarrow{BD}$	(2) Point Plotting Theorem Plane Separation Postulate
(3) E is not in the interior of $\angle ACB$	(3) If E is in the interior of $\angle ACB$ then $\overrightarrow{CE}$ will intersect $\overline{AB}$ (Pasch's theorem) and then $\overrightarrow{CE} \nparallel \overleftrightarrow{AB}$ which contradicts statement 1
(4) E is in the interior of $\angle ACD$	(4) Statements 2 and 3
(5) $m\angle ACD = m\angle ACE + m\angle ECD$	(5) Angle Addition Postulate
(6) $m\angle ACE = m\angle A$	(6) Atlernate interior angles
(7) $\angle ECD \cong \angle B$	(7) Corresponding angles
(8) $m\angle ACD \cong m\angle B + m\angle A$	(8) Statements 5, 6 and 7.

We can take this very theorem as a postulate and prove the angle sum property of a triangle or the Parallel Postulate as a theorem. It may be noted that this theorem is true only in Euclidean geometry. In general, those theorems whose proofs are based either on parallel postulate or any other theorem which is logically equivalent to parallel postulate are true only in Euclidean geometry. The results of such theorems will not be valid in non-Euclidean geometries.

Quadrilaterals and Polygons. The idea of a triangle leads us to the concept of a quadrilateral and then to a polygon in general.

Definition IX (7). (*Quadrilateral*). If A, B, C, D are coplanar points such that :

(1) No three of these four points are collinear.

(2) None of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ intersects any other at a point which is not one of its end points.

Then the union of segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ determine a quadrilateral.

We understand that a quadrilateral is a figure of the form as represented by its physical model in Fig. 134. It has four vertices A, B, C, D. These are coplanar. If the three of the vertices are collinear then the figure will be reduced to a triangle. So any three vertices of a quadrilateral should not be collinear. The second condition has been imposed to exclude the figure of the form Fig. 135. This figure is certainly not a quadrilateral. In this case $\overline{AD}$ and $\overline{BC}$ intersect at O which is not an end point of either of these.

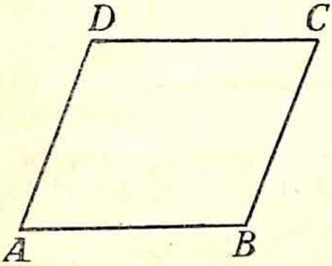


Fig. 134.

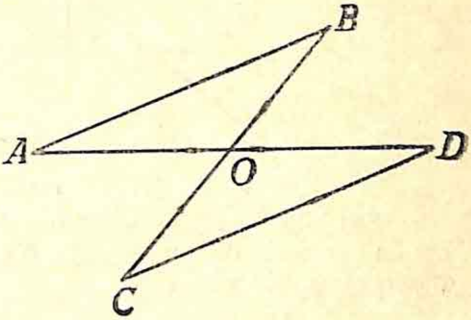


Fig. 135.

As already stated the four points A, B, C, D are called the vertices of a quadrilateral. The four segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ are called the sides of the quadrilateral. The vertices which are the end points of the same side are called consecutive vertices. The vertices which are not consecutive are said to be opposite. Two sides of a quadrilateral which have a common vertex are called consecutive sides. The sides which are not consecutive are called opposite sides. The segments joining the opposite vertices are called diagonals. While naming the quadrilateral by listing its vertices, we agree to follow the following convention.

(1) The vertices that are listed adjacent to each other should be the end points of the same side.

(2) The first and the last vertex listed will be the end points of the same side.

According to this convention, this figure can either be named as ABCD or BCDA or CDAB or DABC or ADCB and so on but not as ACBD or BDCA.

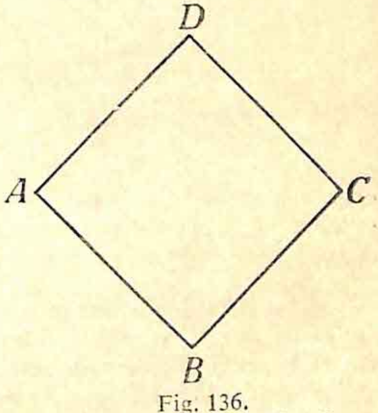


Fig. 136.

Convex Quadrilaterals. In Chapter V, we have defined that a set of points is convex iff (if and only if) any segment whose end points both belong to the given set, lie entirely in the set. If we apply the definition of convexity of a set to the set of points that make the interior of an angle or the interior of a triangle, we find that both the interior of an angle and the interior of a triangle are convex sets. If we take any two points A, B in the interior of an angle or a triangle, then the segment $\overline{AB}$

entirely lies in the interior of the angle or the triangle.

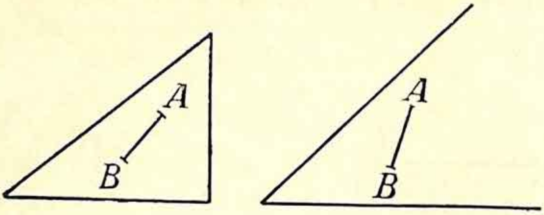


Fig. 137.

But interior of a quadrilateral may not always be a convex set. Union of segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ may determine any one of the quadrilaterals in Fig. 138.

Now the physical model I of quadrilateral ABCD is one whose interior is a convex set as if we take any two points P, Q in its interior, the segment $\overline{PQ}$ lies entirely in the interior of the quadrilateral. It is not the case with the physical model II of the quadrilateral. If we take any two points P, Q in the interior of this set, the segment $\overline{PQ}$ does not entirely lie in its interior. Hence the interior of this quadrilateral is not a convex set.

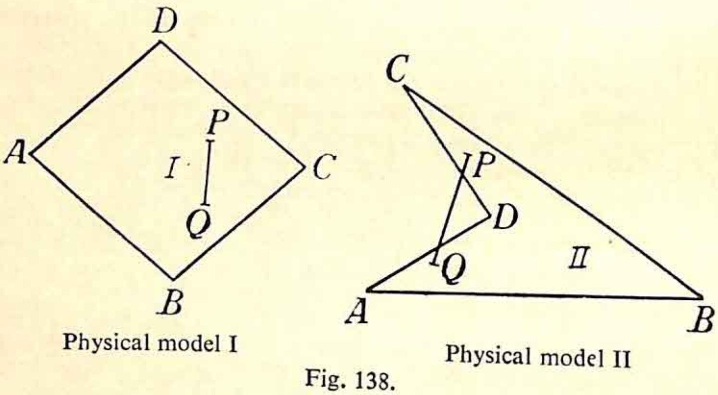


Fig. 138.

It will be seen that in the case of a convex quadrilateral, if any one of its sides ($\overline{AB}$ say) divides the plane of the paper into two half planes H_1 and H_2, then its remaining sides lie in either one of the half planes. We define a convex quadrilateral as follows.

Definition IX (8). (*Convex Quadrilateral*). A quadrilateral is said to be convex iff each of its sides lies in the edge of the half plane which contains the rest of the quadrilateral.

Two angles of a convex quadrilateral are said to be consecutive if these have a side of the quadrilateral in common. The angles which are not consecutive, are said to be opposite.

After having defined a quadrilateral, we define a polygon on the pattern of the definition of a quadrilateral and it is as follows.

Definition IX (9). (*Polygon*). If $P_1, P_2, P_3, \ldots\ldots, P_n$ are n coplanar points with $n \geqslant 3$ and if the n segments $\overline{P_1P_3}$, $\overline{P_2P_3}$, $\overline{P_3P_4}$, ..., $\overline{P_nP_1}$ determined by these points are such that :

(1) No two segments which have a common end point are collinear.

(2) No two segments intersect excepting at their end points.

Then the union of the segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_4}$, ..., $\overline{P_nP_1}$ determine a quadrilateral $P_1P_2P_3, \ldots, P_n$.

The n points $P_1, P_2, P_3, \ldots, P_n$ are called the vertices of the polygon. The n segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_4}$, ..., $\overline{P_nP_1}$ are called the sides of the polygon. A polygon having n vertices has n sides. In the definition of a polygon, condition (1) has been imposed to exclude the possibility of a polygon determined by n vertices having less than n sides. The condition (2) has been imposed to exclude the figures like the one given opposite.

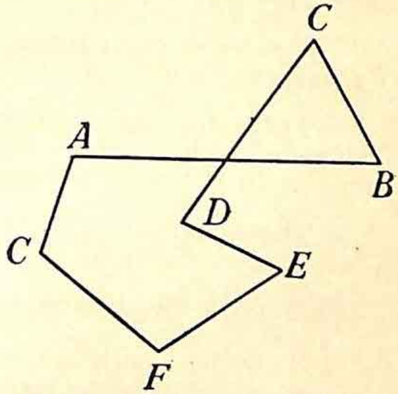


Fig. 139.

The angle determined by any two segments with a common vertex is called the angle of the polygon. Two vertices of the polygon which are the end points of the same side are called consecutive vertices. Two sides of the polygon which have a common end point are called consecutive sides. The segment joining two non-consecutive vertices is called a *diagonal.* We follow the same convention for naming the polygon as we do for a quadrilateral. A polygon can be a convex or it may not be convex. We define a convex polygon as follows.

Definition IX (10). (*Convex Polygon*). A polygon is convex iff each of its sides lies in the edge of the half plane, that contains the rest of the polygon.

A polygon having all its sides congruent to each other and also all its angles congruent to each other is called a regular polygon.

From the definition of a polygon that we have given here, it will be clear that a triangle is a polygon having three vertices, a quadrilateral is a polygon having four vertices etc. A polygon having five vertices is called a pentagon. Following table shows the names of different polygons based on the number of vertices they have :

Vertices	*Name*	*Vertices*	*Name*
3	Triangle	8	Octagon
4	Quadrilateral	9	Nonagon
5	Pentagon	10	Decagon
6	Hexagon	12	Do-decagon
7	Heptagon	15	Penta-decagon
		n	n-gon.

A polygon having all its angles congruent is called an equiangular polygon.

A polygon having all its sides congruent is called an equilateral polygon.

A polygon which is both equiangular and equilateral is called a regular polygon.

The interior of a convex polygon is the intersection of all the half planes each of which has a side of the polygon in its edge and contains the rest of the polygon.

Theorem IX (12). The sum of the measures of angles of a convex polygon of n sides is $(n-2)180$.

To prove this theorem, we prove the following statement.

(1) A triangle is a convex polygon of three sides and sum of the measures of its angles is $(3-2)180$.

(2) The sum of measures of angles of a convex quadrilateral (A convex polygon of four sides) is 360 or $(4-2)180$.

(3) The sum of measures of angles of a convex polygon of 5 sides is $(5-2)180$ etc. etc. and then we generalise and prove that the sum of the measures of angles of a convex polygon of n sides is $(n-2)180$.

Proof.

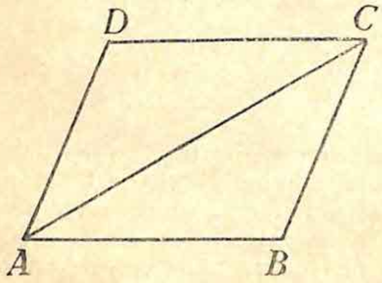


Fig. 140.

Statement 1. We have already proved that the sum of measures of angles of a triangle (A triangle is a convex quadrilateral of three sides—the interior of a triangle being always convex) is 180 which we can write as $(3-2)180$.

Statement 2. The sum of measures of angles of a convex quadrilateral is 360.

Statement	*Reason*
(1) If $\overline{AB}$ divides the plane of the quadrilateral into two half planes then $\overline{BC}$, $\overline{CD}$, $\overline{AD}$ lie in the same half plane	(1) Definition of a convex quadrilateral
(2) D lies on C-side of $\overline{AB}$	(2) $\overline{CD}$ and $\overline{AB}$ do not have any point in common or $\overline{CD} \cap \overline{AB} = \phi$
(3) If $\overline{BC}$ divides the plane of the quadrilateral into two half planes then $\overline{BA}$, $\overline{AD}$, $\overline{CD}$ lie in the same half plane	(3) Definition of a convex quadrilateral
(4) D lies on A-side of $\overline{BC}$	(4) $\overline{AD} \cap \overline{BC} = \phi$
(5) $\overrightarrow{BD}$ lies between $\overrightarrow{BC}$ and $\overrightarrow{BA}$	(5) Statements 2 and 4
(6) $m\angle ABC = m\angle ABD + m\angle DBC$	(6) Angle Addition Postulate
(7) $m\angle ADC = m\angle ADB + m\angle CDB$	(7) Reasoning as for statement 6
(8) $m\angle DAB + m\angle ABD + m\angle ADB = 180$	(8) Angle sum property of a triangle
(9) $m\angle DBC + m\angle BCD + m\angle CDB = 180$	(9) Angle sum property of a triangle
(10) $m\angle DAB + m\angle ABD + m\angle ADB + m\angle DBC + m\angle BCD + m\angle CDB = 360$	(10) Statements 8 and 9
(11) $m\angle DAB + (m\angle ABD + m\angle DBC) + (m\angle ADB + m\angle CDB) + m\angle BCD = 360$	(11) Statement 10
(12) $m\angle DAB + m\angle ABD + m\angle ADC + m\angle BCD = 360$	(12) Statement 11
(13) $m\angle A + m\angle B + m\angle D + m\angle C = 360$	(13) Statement 12.

Statement II. The sum of measures of a convex pentagon is (5−2) 180.

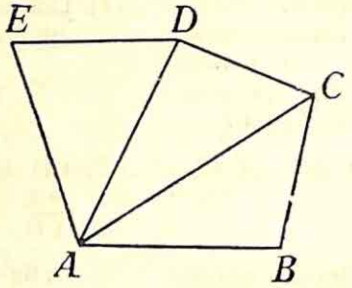


Fig. 141.

Statement	*Reason*
(1) C, D are on E-side of $\overline{AB}$	(1) ABCDE is a Convex Polygon. Hence $\overline{CD} \cap \overline{AB}=\phi$ and $\overline{DE} \cap \overline{AB}=\phi$
(2) C, D are on B-side of $\overline{AE}$	(2) ABCDE is a Convex Polygon etc.
(3) $\overrightarrow{AC}$ and $\overrightarrow{AD}$ lie between $\overrightarrow{AB}$ and $\overrightarrow{AE}$	(3) Statements 1 and 2
(4) $\overrightarrow{AC}$ lies between $\overrightarrow{AB}$ and $\overrightarrow{AD}$	(4) Reasoning as for statement 3
(5) $m\angle A = m\angle EAD + m\angle DAC + m\angle CAB$	(5) Angle Addition Postulate
(6) $m\angle C = m\angle BCA + m\angle ACD$	(6) Reasoning similar as for statement 3
(7) $m\angle D = m\angle CDA + m\angle ADE$	(7) Reasoning similar as for statement 3
(8) $m\angle CAB + m\angle ABC + m\angle BCA = 180$	(8) Angle-sum property of a triangle
(9) $m\angle ACD + m\angle CDA + m\angle DAC = 180$	(9) Angle-sum property of a triangle
(10) $m\angle ADE + m\angle DEA + m\angle EAD = 180$	(10) Angle-sum property of a triangle
(11) $m\angle A + m\angle B + m\angle C + m\angle D + m\angle E = 540$	(11) Statements 5, 6, 7, 8, 9, 10
(12) $m\angle A + m\angle B + m\angle C + m\angle D + m\angle E = (5-2)\ 180$	(12) Statement 11.

We have proved that :

(*a*) The sum of measures of angles of a Convex Polygon of 3 sides is $(3-2)$ 180.

(*b*) The sum of measures of angles of a Convex Polygon of 4 sides is $(4-2)$ 180.

(*c*) The sum of measures of angles of a Convex Polygon of 5 sides is $(5-2)$ 180.

We can generalise and assert that the sum of measures of angles of a convex polygon of n sides is $(n-2)$ 180.

Mathematically speaking this proof is not very rigorous but it can very well serve the purpose of this course.

Corollary 1. The measure of each interior angle of a regular polygon of n-sides is $\frac{(n-2)\ 180}{n}$.

Corollary 2. The sum of the measures of exterior angles of a polygon of any number of sides formed by extending each of its sides in succession is 360.

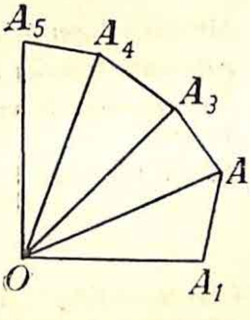


Fig. 142.

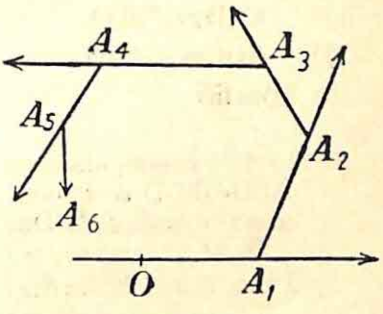


Fig. 143.

Hint. At each of the n-vertices O, A_1, A_2, $A_3, \ldots, A_{n-1}$, the sum of the measures of an angle (interior angle) of a polygon and its adjacent interior angle is 180. Hence the sum of the measures of all the exterior and interior angles is $n \times 180$. We have already proved that the sum of the measures of angles of the polygon (interior angles) is $(n-2)$ 180. Hence the sum of the measures of exterior angles is $n \times 180-(n-2)\ 180=2\times 180=360$.

Parallelogram. Now we define a special and a very important type of quadrilateral which is known as a parallelogram. Before we do so, we review our definition of parallel segments which states that two segments are said to be parallel iff the lines that contain them are parallel.

Definition IX (11). (*Parallelogram*). A quadrilateral having each of its sides parallel to the side opposite to it, is known as a parallelogram. A parallelogram is denoted by $\|^{gm}$.

The definition of a $\|^{gm}$ points to the fact that a $\|^{gm}$ is a convex quadrilateral. After having defined a $\|^{gm}$, we prove the following very important theorems.

Theorem IX (13). Opposite sides of any $\|^{gm}$ are congruent.

Given : $\|^{gm}$ ABCD in which $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{AD}$.

To prove that : (*i*) $\overline{AB} \cong \overline{CD}$ (*ii*) $\overline{BC} \cong \overline{AD}$.

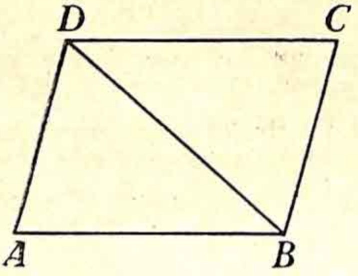


Fig. 144.

Proof.

Statement	*Reason*
(1) $\angle CBD \cong \angle BDA$	(1) Alternate Interior Angles
(2) $\angle BDC \cong \angle ABD$	(2) Alternate Interior Angles
(3) $\overline{BD} \cong \overline{BD}$	(3) Any segment is congruent to itself
(4) 1—1 correspondence ABD$\leftrightarrow$BCD is a congruence and $\triangle ABD \cong \triangle BCD$	(4) A.S.A. theorem
(5) $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$	(5) Corresponding parts of congruent triangles.

Theorem IX (14). Opposite angles of any parallelogram are congruent.

Proof. In theorem IX (13), we proved that $\triangle ABD \cong \triangle BCD$ which implies that $\angle A \cong \angle C$. By proving $1-1$ correspondence ACB$\leftrightarrow$CAD a congruence and $\triangle ACB \cong \triangle CAD$, we can prove that $\angle B \cong \angle D$.

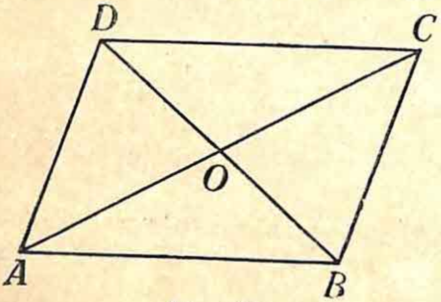


Fig. 145.

Theorem IX (15). The diagonals of a parallelogram bisect each other.

Given : $\|^{gm}$ ABCD. Diagonals $\overline{AC}$, $\overline{BD}$ have a common point O. ($\overline{AC} \nparallel \overline{BD}$. Why ?)

To prove that : (*i*) $\overline{OA} \cong \overline{OC}$ (*ii*) $\overline{OB} \cong \overline{OD}$.

Proof.

Statement	*Reason*
(1) $\angle OAB \cong \angle OCD$	(1) Alternate Interior Angles
(2) $\angle OBA \cong \angle ODC$	(2) Alternate Interior Angles
(3) $\overline{AB} \cong \overline{CD}$	(3) Theorem IX (13)
(4) 1—1 correspondence $OAB \leftrightarrow OCD$ is a congruence and $\triangle OAB \cong \triangle OCD$	(4) A.S.A. theorem
(5) $\overline{OA} \cong \overline{OC}$ and $\overline{OB} \cong \overline{OD}$	(5) Corresponding parts of congruent triangles.

Theorem IX (16). If opposite sides of any quadrilateral are congruent then the quadrilateral is a $\parallel^{gm}$.

Given : A quadrilateral ABCD in which $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$.

To prove that : Quadrilateral ABCD is a $\parallel^{gm}$.

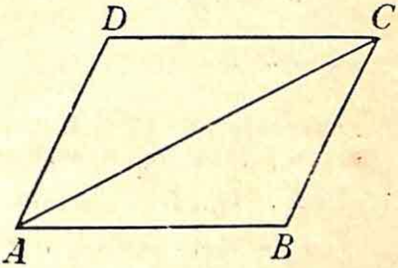


Fig. 146.

Proof.

Statement	*Reason*
(1) $\overline{AB} \cong \overline{CD}$; $\overline{BC} \cong \overline{AD}$	(1) Given
(2) $\overline{AC} = \overline{CA}$	(2) Any segment is congruent to itself
(3) 1—1 correspondence $ABC \leftrightarrow CDA$ is a congruence and $\triangle ABC \cong \triangle CDA$	(3) S.S.S. Theorem
(4) $\angle BAC \cong \angle ACD$ and $\angle BCA \cong \angle CAD$	(4) Corresponding parts of congruent triangles
(5) $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{AD}$	(5) Statement 4 ; As these are pairs of alternate interior angles.

This theorem is converse to Theorem IX (13).

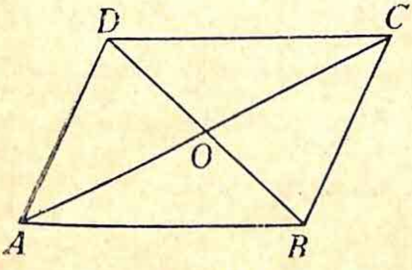


Fig. 147.

Theorem IX (17). [*Converse to theorem IX (15)*]. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a $\parallel^{gm}$.

Given : A quadrilateral ABCD. The diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other at O. ($\overline{AO} \cong \overline{OC}$ and $\overline{BO} \cong \overline{OD}$).

To prove that : Quadrilateral ABCD is a $\parallel^{gm}$.

Proof.

Statement	*Reason*
(1) $\overline{AO} \cong \overline{OC}$ and $\overline{BO} \cong \overline{OD}$	(1) Given
(2) $\angle AOB \cong \angle DOC$	(2) Vertical angles
(3) 1 — 1 correspondence AOB$\leftrightarrow$COD is a congruence and $\triangle AOB \cong \triangle COD$	(3) S.A.S. Postulate
(4) $\overline{AB} \cong \overline{DC}$	(4) Corresponding parts of congruent triangles
(5) $\overline{BC} \cong \overline{AD}$	(5) By similar reasoning as for statement 4
(6) ABCD is a $\|^{gm}$.	(6) Statements 4 and 5; Theorem IX (16).

Theorem IX (18). If two sides of a quadrilateral are congruent and parallel, the quadrilateral is a $\|^{gm}$.

Given : Quadrilateral ABCD in which $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$.

To prove that : ABCD is a $\|^{gm}$.

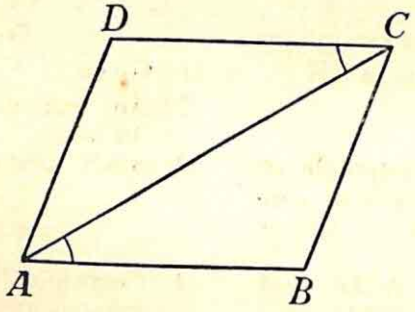


Fig. 148.

Proof.

Statement	*Reason*
(1) $\overline{AB} \cong \overline{CD}$	(1) Given
(2) $\overline{AB} \parallel \overline{CD}$	(2) Given
(3) $\angle BAC \cong \angle DCA$	(3) Alternate interior angles
(4) $\overline{AC} \cong \overline{CA}$	(4) Any segment is congruent to itself
(5) 1 — 1 correspondence BAC$\leftrightarrow$DCA is a congruence and $\triangle BAC \cong \triangle DCA$	(5) S.A.S. Postulate

(6) $\overline{BC} \cong \overline{AD}$	(6) Corresponding parts of congruent triangles
(7) Quadrilateral ABCD is a $\|^{gm}$.	(7) Theorem IX (16).

Definition IX (12). (*Rhombus*) :—A $\|^{gm}$ which has two adjacent sides congruent is called a rhombus.

Since the set of rhombuses is a subset of set of parallelograms, every theorem that has been proved for the set of parallelograms will also be true in the case of a rhombus. From the definition it is clear that measures of all sides of a rhombus are equal to each other.

Definition IX (13). (*Rectangle*). A $\|^{gm}$ having one of its angles a right angle is called a rectangle.

From the definition it is clear that the measure of each angle of a rectangle will be 90.

Definition IX (14). (*Square*). A rectangle having two adjacent sides congruent is called a square.

From these definitions we can state the following :

(1) The set of squares is a subset of set of rectangles.

(2) The set of rectangles is a subset of set of parallelogrames.

(3) The set of squares is a subset of set of rhombuses.
Further we can state that :

(*a*) Measure of all the sides of a rhombus is same.

(*b*) All the angles of a rectangle are right angles.

(*c*) All the angles of a square are right angles.

(*d*) Measure of all the sides of a square is same.

The reader is advised to prove these results with the help of theorems that we have proved for parallelograms.

A square or a rhombus can be called an equilateral $\|^{gm}$ while a square or a rectangle can be called an equiangular $\|^{gm}$.

Theorem IX (19). The diagonals of a rhombus are perpendicular to each other.

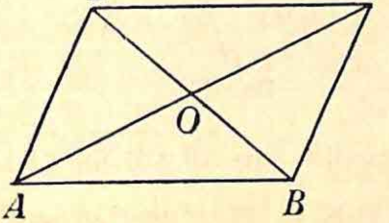


Fig. 149.

Given : A rhombus ABCD. Diagonals $\overline{AC}$ and BD intersect at O,

To prove that : $\overline{AC} \perp \overline{BD}$.

Proof.

Statement	*Reason*
(1) $\overline{AB} \cong \overline{BC}$	(1) Given. ; ABCD is a rhombus
(2) $\overline{AO} \cong \overline{OC}$	(2) ABCD is a $\parallel^{gm}$.
(3) $\overline{OB} \cong \overline{OB}$	(3) Any segment congruent to itself
(4) 1—1 correspondence AOB $\leftrightarrow$COB is a congruence and $\triangle AOB \cong \triangle COB$	(4) S.S.S. Theorem
(5) $\angle AOB \cong \angle BOC$	(5) Corresponding parts of congruent triangles
(6) $m\angle AOB + m\angle BOC = 180$	(6) $\angle AOB$, $\angle BOC$ form a linear pair
(7) Measure of each of $\angle AOB$, $\angle BOC$ is 90	(7) Statements 5 and 6
(8) $\overline{BD} \perp \overline{AC}$	(8) Statement 7.

Theorem IX (20). If three or more parallel lines intercept congruent segments on one transversal, then they intercept congruent segments on any other transversal.

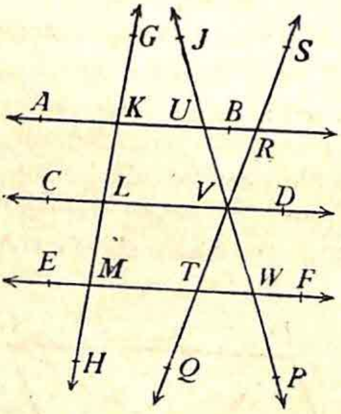


Fig. 150.

Given : Three parallel lines $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ and a transversal $\overleftrightarrow{GH}$ intersecting them at K, L, M respectively such that $\overline{KL} \cong \overline{LM}$. Any other transversal $\overleftrightarrow{JP}$ intersects them in U, V, W respectively.

To prove that : $\overline{UV}\cong\overline{VW}$.

Proof.

Statement	*Reason*
(1) There exists a line $\overleftrightarrow{QS}\parallel\overleftrightarrow{GH}$ through V	(1) Parallel Postulate
(2) $\overleftrightarrow{QS}$ intersects $\overleftrightarrow{AB}$ in R and $\overleftrightarrow{EF}$ in T	(2) Through any point, there is atmost one line parallel to a given line and excepting the given line it must intersect every other line in the same plane
(3) Quad. LVRK is a parallelogram	(3) $\overleftrightarrow{KR}\parallel\overleftrightarrow{LV}$ (given) and $\overleftrightarrow{KL}\parallel\overleftrightarrow{RV}$ (Statement 1)
(4) $\overline{VR}\cong\overline{KL}$	(4) Opposite sides of a parallelogram
(5) $\overline{VT}\cong\overline{LM}$	(5) Reasoning similar to the one for statement 4
(6) $\overline{KL}\cong\overline{LM}$	(6) Given
(7) $VR\cong\overline{VT}$	(7) Statements 4, 5, 6
(8) $\angle URV\cong\angle VTW$	(8) Alternate Interior Angles
(9) $\angle UVR\cong\angle TVW$	(9) Vertical angles
(10) 1—1 Correspondence URV $\leftrightarrow$WTV is a congruence and $\triangle URV\cong\triangle WTV$	(10) A.S.A. Theorem
(11) $\overline{UV}\cong\overline{VW}$	(11) Corresponding parts of congruent triangles.

Above we have proved the theorem in the case when the transversals GH and JP are not parallel to each other. It is possible, of course, that these may be parallel. In such a case KLVU will be a parallelogram and hence $\overline{KL}\cong UV$. Also LMWV will be a parallelogram and hence $\overline{LM}\cong\overline{VW}$. As we are given that $\overline{KL}\cong\overline{LM}$ so we arrive at the result that $\overline{UV}\cong\overline{VW}$.

Another possibility is that the transversals $\overline{GH}$ and $\overline{JP}$ meet at K (as shown in Fig. 151). We can proceed to prove this case in the same manner as we proved the first case.

Yet another case will be that when the two transversals meet at L (as shown in Fig. 152). In this case we will prove that 1—1 correspondence ULK↔WLM is a congruence and $\triangle ULK \cong \triangle WLM$ as $\overline{KL} \cong \overline{LM}$ (given) and $\angle ULK \cong \angle WLM$ (vertical angles) and $\angle UKL \cong \angle WML$ (alter nate interior angles). Hence we prove that $\overline{UL} \cong \overline{LW}$ or $\overline{UV} \cong \overline{VW}$.

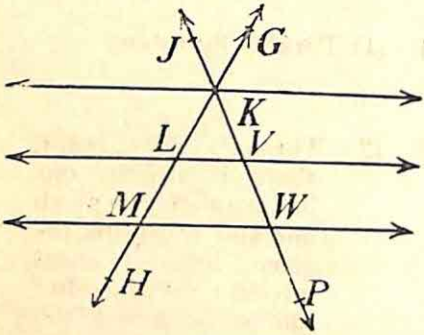


Fig. 151.

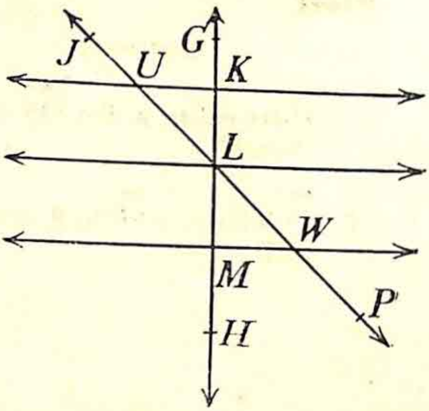


Fig. 152.

Problem Set

A

(1) Prove that if the lines p, q, r are coplanar and if $p \parallel q$ and $q \parallel r$ then $p \parallel r$ (Transitivity relation of parallelism between two parallel lines).

Given : Coplanar lines p, q, r such that $p \parallel q$ and $q \parallel r$.

To prove that : $p \parallel r$.

Proof. (*i*) Assume $p \nparallel r$. Then p and r intersect. Now $p \parallel q$ and $r \parallel q$ and p and r intersect at a point A, say. It implies that through A, the point of intersection of p and r, two lines p and r are parallel to the same line q which contradicts the parallel postulate. Hence the assumption that $p \nparallel r$ has to be abandoned. Hence the only other alternative that $p \parallel r$ has to be accepted.

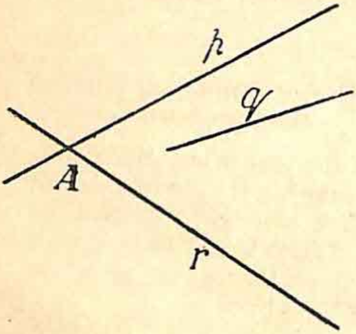


Fig. 153.

(2) If a line intersects one of two parallel lines, it intersects the other. (It immediately follows from the result of problem 1. If $p \parallel q$ and r and p intersect at A, then $r \nparallel q$ as otherwise there will be two lines through A parallel to the same line q.)

(3) In a plane, any two lines that are parallel to the third line are parallel to one another.

($p \parallel r$ and $q \parallel r$. Assume $p \nparallel q$. Then through the point of intersection of p, q two lines p and q are parallel to the same line r.)

(4) Through a given point, there can be only one line parallel to a given line. [Through a point, there exists atleast one line $\parallel$ to a give line. (Why ?) Also there is atmost one line through a point parallel to a given line. (Why ?)]

(5) A ray $\overrightarrow{AB}$ is parallel to a line p and also a ray $\overrightarrow{AC}$ is parallel to the same line p. Prove that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays. Given that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are distinct rays.

B

(1) If two coplanar lines are intersected by a transversal such that a pair of alternate exterior angles are congruent, then the lines are parallel.

[In this figure (1, 8) and (2, 7) are the two pairs of alternate exterior angles.]

(2) If each of the two coplanar lines is perpendicular to the same line then the two lines ars parallel.

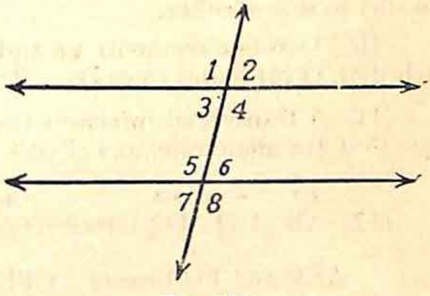


Fig. 154.

(3) In this figure given that $\angle ABC \cong \angle BCD$; $\overrightarrow{BE}$ is the angle bisector of $\angle ABC$ and

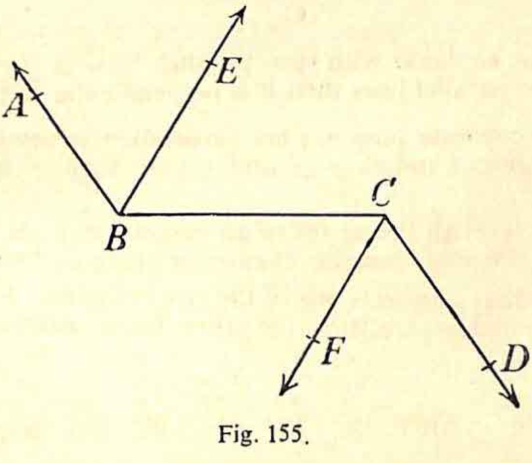


Fig. 155.

$\overrightarrow{CF}$ is the angle bisector of $\angle BCD$. Prove that (*i*) $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ (*ii*) $\overleftrightarrow{BE} \parallel \overleftrightarrow{CF}$.

(4) If two coplanar lines form interior angles with a transversal which are equal in measure, the lines are parallel.

(5) If two coplanar lines form corresponding angles with a transversal which are equal in measure, the lines are parallel.

(6) If two coplanar lines form supplementary interior angles on the same side of a transversal, the lines are parallel.

(7) Write any five ways of proving two lines to be parallel.

(8) If two coplanar lines form a pair of supplementary exterior angles on the same side of a transversal, the lines are parallel. [In the figure of problem 1, (2, 8) and (1, 7) are the two pairs of exterior angles on the same side of a transversal.]

(9) The lines which are perpendicular to the same line are parallel to one another.

(10) Two line segments $\overline{AB}$ and $\overline{CD}$ have a common point E such that $\overline{AE} \cong \overline{EB}$ and $\overline{CE} \cong \overline{ED}$. Prove that $\overline{AC} \parallel \overline{DB}$ and $\overline{AD} \parallel \overline{CB}$.

(11) A transversal intersects two parallel lines. Prove that the lines that are angle bisectors of pair of alternate angles are parallel.

(12) $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. $\overleftrightarrow{PQ}$ intersects $\overleftrightarrow{AB}$ in E and $\overleftrightarrow{CD}$ in F. $\overrightarrow{EG}$ bisects angle $\angle AEF$ and $\overrightarrow{FG}$ bisects $\angle CFE$. $\overrightarrow{EG}$ and $\overrightarrow{FG}$ have a point G in common. Prove that $m\angle EGF = 90$.

C

(1) If a line coplanar with two parallel lines is perpendicular to one of the two parallel lines then it is perpendicular to the other.

(2) If two coplanar lines p, q are perpendicular respectively to two intersecting lines l and m ($p \perp l$ and $q \perp m$) then p and q will intersect.

(3) A line through the vertex of an isosceles triangle parallel to the base will be the angle bisector of exterior angle at the vertex.

(4) A line that intersects one of the two congruent sides of an isosceles triangle and is parallel to the other, forms another isosceles triangle.

(5) Given a $\triangle ABC$. $\overrightarrow{BD} \perp \overleftrightarrow{BC}$. $\overrightarrow{CE} \parallel \overleftrightarrow{BD}$. $\overrightarrow{BA}$ is the angle bisector of $\angle DBC$ and $\overrightarrow{CA}$ is the angle bisector of $\angle ECB$. Prove that $\triangle ABC$ is isosceles.

(6) Prove that if two coplanar angles have their sides respectively parallel then the angles are congruent.

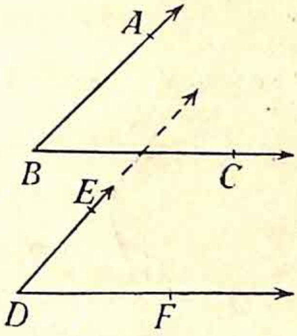


Fig. 156.

(7) Prove that if two coplanar angles have their sides respectively anti-parallel, then the angles are conguent.

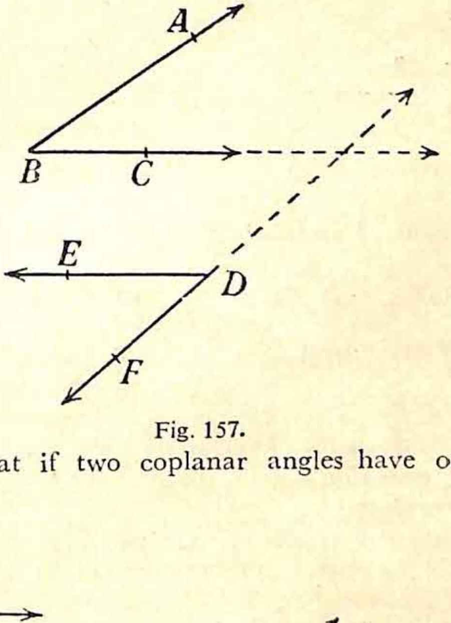


Fig. 157.

(8) Prove that if two coplanar angles have one side of each

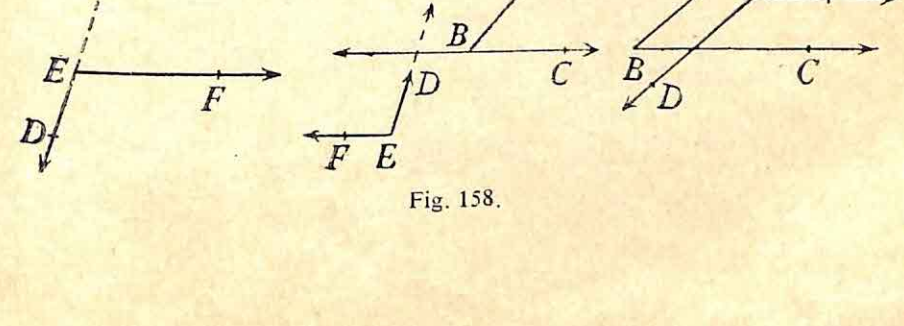


Fig. 158.

parallel while the other side of each is anti-parallel then the given angles are supplementary.

(9) Assume theorem IX(3) as a postulate and prove the parallel postulate as a theorem.

Given : (1) Two lines p, q having a common point A and the line p is parallel to the other line r. (2) Alternate interior angles so formed are congruent.

To prove that : $q \nparallel r$.

Proof : Let $\overleftrightarrow{EF}$ be the transversal containing A. Let this transversal divide the plane containing these lines into two half planes H_1 and H_2.

Let $C \in p$ be in H_1 and $D \in r$ be in H_2.

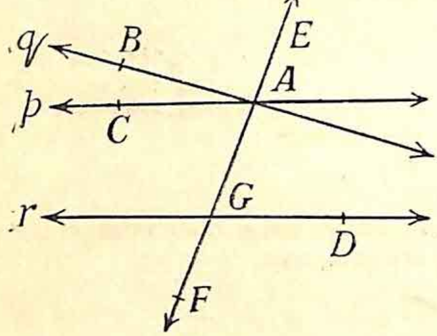


Fig. 159.

Statement	*Reason*
(1) $p \parallel r$	(1) Given
(2) $\angle CAG \cong \angle DGA$	(2) Alternate interior angles
(3) $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are distinct rays	(3) p and q are distinct lines.
(4) $m\angle CAG \neq m\angle BAG$	(4) Angle construction postulate.
(5) $m\angle BAG \neq m\angle DGA$	(5) Statements 2 and 4
(6) $q \nparallel r$	(6) Theorem IX/3 which has been assumed as postulate.

In fact we should have assumed $q \parallel r$ and then we should have proved that it leads to a contradiction of the theorem IX(3) which we assume as a postulate.

(10) Assume the result of problem 1 (If a line lies in the plane of two parallel lines and is perpendicular to one of them then it is perpendicular to the other) as a postulate and prove the parallel postulate as a theorem.

(11) Given an angle $\angle ABC$. $\overrightarrow{BR}$ is the angle bisector of $\angle ABC$. $P \in \overrightarrow{BR}$. $\overrightarrow{PX} \parallel \overrightarrow{BC}$ have a point D in common with $\overrightarrow{BA}$. Prove that $\triangle BPD$ is isosceles.

D

(1) Given 1—1 correspondence between the vertices of two triangles ABC↔DEF. If two pairs of corresponding angles are congruent then the third pair of corresponding angles is also congruent.

(2) $\triangle ABC$ is a right $\triangle$ with $m\angle A=90$. Prove that $\angle B$ and $\angle C$ are complimentary.

(3) Can you prove the angle-sum property of a triangle without making use of parallel postulate ? If not, why not ?

(We use parallel postulate in proving alternate interior angles congruent. Also it assumes that there is one and only line through C parallel to base $\overline{AB}$).

(4) Can you prove the strong form of exterior angle theorem without making use of parallel postulate ? If not, why not ?

(5) The bisector of an exterior angle at the vertex of any triangle is parallel to the base iff the triangle is isosceles. (The use of 'iff' in the statement implies that both the theorem and its converse is to be proved.)

(6) Given a right $\triangle ABC$ with $m\angle A=90$. $D \in \overline{BC}$ such that $\overline{DA} \cong \overline{DB}$, prove that $\overline{DB} \cong \overline{DC}$.

(7) Given a right $\triangle ABC$ with $m\angle A=90$. $D\in\overline{AB}$, $E \in \overline{BC}$ and $F \in \overline{CF}$ such that A—D—B, B—E—C and C—F—A and $\overline{BD}\cong\overline{BE}$ and $\overline{CE} \cong \overline{CF}$, prove that $m\angle DEF=45$.

(8) Given a $\triangle ABC$ with $\angle C$ an obtuse angle. Prove that the foot of perpendicular from A to $\overleftrightarrow{BC}$ is not a point of segment $\overline{BC}$. [Assume $D \in \overline{BC}$. If $D=B$, then $m\angle B=90$ and $m\angle C>90$ (given). Then the sum of measures of angles of $\triangle ABC$ is more than 180 which contradicts angle-sum property of a triangle. Now let $D\in\overline{BC}$ such that B—D—C, then $m\angle ADB=90$ and then $m\angle ADB<m\angle C$ and this contradicts the weak form of exterior angle theorem. Now assume $D=C$. It contradicts the hypothesis that $\angle C$ is obtuse.]

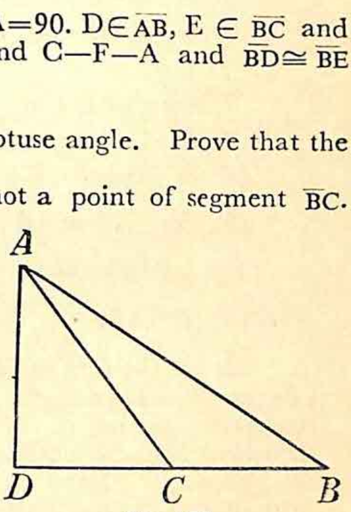


Fig. 160.

(9) Without using angle-sum property of a triangle, prove that if $\triangle ABC$ and $\triangle DEF$ have $\overline{AB} \cong \overline{DE}$, $\angle ABC\cong\angle DEF$ and $\angle BCA\cong\angle EFD$ then $\triangle ABC\cong\triangle DEF$.

(10) Prove that in a triangle, there can be only one right angle or one obtuse angle.

(11) If two right triangles have the hypotenuse and one acute angle of the one congruent respectively to the hypotenuse and one acute angle of the other, the triangles are congruent.

(12) Prove that in an isosceles triangle the altitudes on congruent sides are equal in measure.

(13) The perpendiculars from any point in the bisector of an angle to the sides of the angle are equal in measure.

(14) Perpendiculars from any point in the base of an isosceles triangle to the congruent sides make angles with the base which are equal in measure.

(15) Following is a proof for angle-sum property of a triangle.

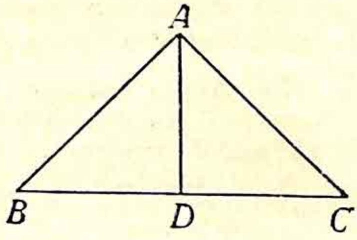


Fig. 161.

Proof: Assume that the sum of the measures of angles of any triangle is constant and equals k.

(1) $m\angle A + m\angle B + m\angle C = k$.
(2) $m\angle BAD + m\angle B + m\angle ADB = k$.
(3) $m\angle CAD + m\angle C + m\angle ADC = k$.
(4) $m\angle BAD + m\angle CAD = m\angle A$.
(5) $m\angle ADB + m\angle ADC = 180$.
(6) $m\angle BAD + m\angle B + m\angle ADB + m\angle CAD + m\angle C + m\angle ADC = 2k$.
(7) $(m\angle BAD + m\angle CAD) + m\angle B + (m\angle ADB + m\angle ADC) + m\angle C = 2k$.
(8) $m\angle A + m\angle B + 180 + m\angle C = 2k$.
(9) $(m\angle A + m\angle B + m\angle C) + 180 = 2k$.
(10) $k + 180 = 2k$.
(11) $180 = k$.

Hence the sum of measures of angles of a triangle is 180. In this proof we have made use of some assumptions which you are required to list down. Have we made use of parallel postulate or any statement logically equivalent to parallel postulate in the proof of this theorem ? Have we directly or indirectly made use of strong form of exterior angle theorem ?

(16) If the bisector of an angle of a triangle and the perpendicular bisector of the opposite side do not coincide, prove that their intersection is always a point in the exterior of the triangle. (This problem tells the flaw in our proof of the theorem : 'Prove that every triangle is isosceles' given in the first chapter in which although the bisector of $\angle A$ and the right bisector of $\overline{BC}$ do not coincide, yet we have taken their intersection to be in the interior of the $\triangle$.)

Hint. Consider any arbitrary triangle which is scalene. There are three alternatives for the point of intersection of the bisector of the angle and the right bisector of opposite side. (*i*) Either these intersect in the interior of the $\triangle$ or (*ii*) on the side opposite the angle or (*iii*) on the opposite side. Reject the first two alternatives as these lead to the conclusion that the triangle that is being considered is isosceles.

(17) Given a $\triangle ABC$. D is a point on $\overleftrightarrow{BC}$ such that D–B–C. E is a point on $\overleftrightarrow{AB}$ such that E–A–B. F is a point on $\overleftrightarrow{AC}$ such that A–C–F. Prove that $m\angle EAC + m\angle DBA + m\angle FCB = 360$.

(18) The angle bisectors of $\angle B$ and $\angle C$ of a $\triangle ABC$ meet in O which lies in the interior of $\triangle ABC$. Prove that $m\angle BOC = 90$.

(19) Given a $\triangle ABC$. D is a point on $\overleftrightarrow{BC}$ such that B–C–D. $\overrightarrow{BE}$ is the angle bisector of $\angle B$ and $\overrightarrow{CF}$ is the angle bisector of $\angle ACD$. $\overrightarrow{BE}$ and $\overrightarrow{CF}$ has a common point P. Prove that $m\angle BPC = \frac{1}{2}$ $m\angle A$.

(20) Given a $\triangle ABC$. D is a point on $\overleftrightarrow{AB}$ such that A–B–D. E is a point on $\overleftrightarrow{AC}$ such that A–C–E. $\overrightarrow{BF}$ is the bisector of $\angle CBD$ and $\overrightarrow{CG}$ is angle bisector of $\angle ECB$. $\overrightarrow{BF}$ and $\overrightarrow{CG}$ have a point P in common. Prove that $m\angle BPC = 90 - \frac{1}{2}\, m\angle A$.

(21) Given a $\triangle ABC$. $\overrightarrow{AD}$ is the angle bisector of $\angle A$ and $\overline{AE} \perp \overline{BC}$. Prove that $m\angle EAD = \frac{1}{2}(m\angle B - m\angle C)$ if $m\angle B > m\angle C$ and $m\angle EAD = \frac{1}{2}\,(m\angle C - m\angle B)$ if $m\angle B < m\angle C$.

(22) The sides of an n-gon are produced to form a star shaped figure. Prove that the sum of measures of angles at the vertices is $2(n-4)$.

(23) Find the number of degrees in each interior angle of a (*i*) regular pentagon (*ii*) regular hexagon (*iii*) regular heptagon (*iv*) regular octagon (*v*) regular decagon (*vi*) regular polygon of 20 sides.

(24) As the number of sides of a regular polygon, does the measure of exterior angle increase or decrease ? What about the measure of an interior angle ? What is the measure of smallest interior angle that an equiangular polygon may have ?

(25) Find the number of sides of a polygon if the sum of measures of its interior angle is twice the sum of the measures of its exterior angles.

(26) ABCDE is a regular pentagon. $\overline{DF} \perp \overline{AB}$ such that A−F−B. Prove that F is mid-point of $\overline{AB}$.

(27) Prove that the opposite sides of a regular hexagon are parallel.

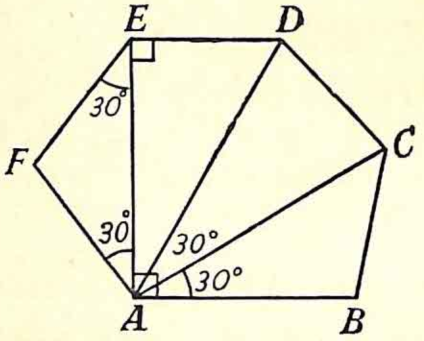


Fig. 162.

E

(1) Prove that two parallel lines are everywhere equidistant.

(2) Prove that if the diagonals of a $\parallel^{gm}$ are perpendicular to each other, the $\parallel^{gm}$ is equilateral.

(3) Prove that every pair of consecutive angles of a $\parallel^{gm}$ are supplementary.

(4) Prove that if the diagonals of a $\parallel^{gm}$ bisect every angle (angles at the vertices that each diagonal joins) then the $\parallel^{gm}$ is equilateral.

(5) The joins of mid. points of opposite sides of a $\parallel^{gm}$ divides it into four congruent parallelograms. (Write your own definition of congruent parallelograms on the lines of definition of congruent triangles.)

(6) If one pair of opposite sides of a quadrilateral is parallel to each other while the other pair is congruent to each other, will this quadrilateral be a $\parallel^{gm}$? Give arguments in support of your answer. Prove that if a pair of opposite sides of a quadrilateral are congruent as well as parallel the quadrilateral is a $\parallel^{gm}$.

(7) Prove that if the diagonals of a $\parallel^{gm}$ are congruent then the $\parallel^{gm}$ is a rectangle.

(8) The joins of mid. points of four sides of a $\parallel^{gm}$ determine another $\parallel^{gm}$.

(9) If the diagonals of a rectangle are perpendicular to each other, then the rectangle is a square.

(10) The angle bisectors of the angles of a $\parallel^{gm}$ (if it is not a rhombus) determine a rectangle. What happens when the $\parallel^{gm}$ is a rhombus ?

(11) The segment determined by the mid. points of two sides of a triangle is parallel to the third side and half as long as the third side.

Given : $\triangle ABC$. D is mid. point of $\overline{AB}$ and E is mid. point of $\overline{BC}$.

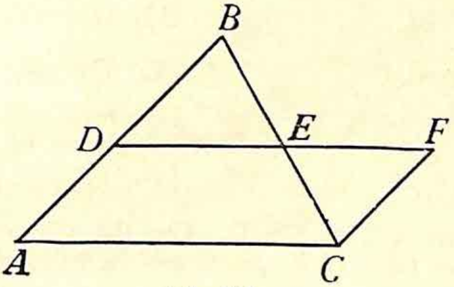


Fig. 163.

To prove that : (*i*) $\overline{DE} \parallel \overline{AC}$ (*ii*) $m\overline{DE} = \frac{1}{2}\, m\overline{AC}$.

Proof.

Statement	*Reason*
(1) There exists a point F on $\overrightarrow{DE}$ such that $\overline{DE} \cong \overline{EF}$	(1) Point plotting theorem
(2) $\overline{BE} \cong \overline{EC}$	(2) Given
(3) $\angle BED \cong \angle FEC$	(3) Vertical angles
(4) 1—1 correspondence $BED \leftrightarrow CEF$ is a congruence and $\triangle BED \cong \triangle CEF$	(4) S.A.S. Postulate
(5) $\angle BDE \cong \angle EFC$	(5) Corresponding parts of congruent triangles
(6) $\overleftrightarrow{BD} \parallel \overleftrightarrow{FC}$	(6) Statement 5 ; As alternate interior angles are congruent
(7) $\overline{BD} \cong \overline{FC}$	(7) Statement 4; Corresponding parts of congruent triangles
(8) $\overline{BD} \cong \overline{DA}$	(8) Given

(9) $\overline{FC} \cong \overline{DA}$	(9) Statements 7 and 8
(10) $\overleftrightarrow{DA} \parallel \overleftrightarrow{FC}$	(10) Statement 6 ; $\overleftrightarrow{BD} = \overleftrightarrow{DA}$
(11) DACF is a $\parallel^{gm}$	(11) If a pair of opposite sides of a quadrilateral are congruent as well as parallel then it is a $\parallel^{gm}$; Theorem IX (18)
(12) $\overline{DF} \parallel \overline{AC}$	(12) Opposite sides of a $\parallel^{gm}$
(13) $\overline{DF} \cong \overline{AC}$	(13) Opposite sides of a $\parallel^{gm}$
(14) $m\overline{DE} = \frac{1}{2} m\overline{AC}$	(14) $m\overline{DE} = \frac{1}{2}\, m\overline{DF}$; Statement 12
(15) $\overline{DE} \parallel \overline{AC}$	(15) Statement 12.

(12) Given a $\parallel^{gm}$. ABCD. On the diagonal $\overline{AC}$, there are points E, F such that A–E–F–C and $\overline{AE} \cong \overline{FC}$. Prove that DEBF is a $\parallel^{gm}$.

(13) Given $\triangle$ABC and $\triangle$DEF such that $\overline{AD} \parallel \overline{EC}$ and $\overline{EC} \parallel \overline{FB}$. Also $\overline{AC} \parallel \overline{DE}$ and $\overline{CB} \parallel \overline{EF}$. Prove that ABFD is a $\parallel^{gm}$.

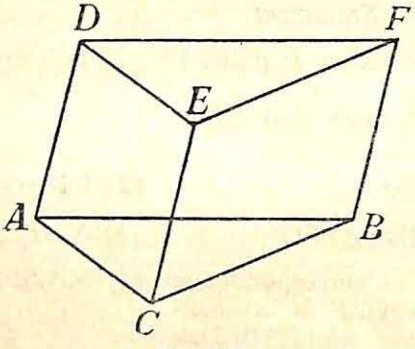


Fig. 164.

Hint. Prove that ACED is a $\parallel^{gm}$. ECBF is a $\parallel^{gm}$. and hence $\overline{AD} \cong \overline{EC}$ and $\overline{EC} \cong \overline{FB}$. Therfeore, $\overline{AD} \cong \overline{FB}$. Also $\overline{AD} \parallel \overline{FB}$.

(14) Given an isosceles $\triangle$ABC with $\overline{AB} \cong \overline{AC}$. D is a point in $\overline{BC}$ such that B–D–C. E is a point in $\overline{AB}$ such that A–E–B and $\overline{DE} \parallel \overline{AC}$. F is a point in $\overline{AC}$ such that A–F–C and $\overline{DF} \parallel \overline{AB}$. Prove that AEDF is a $\parallel^{gm}$.

(15) ABC is an isosceles triangle with $\overline{AB} \cong \overline{AC}$. D, E, F are the mid. points of $\overline{BC}$, $\overline{AB}$, $\overline{AC}$ respectively. Prove that AEDF is a rhombus.

(16) Given a $\parallel^{gm}$ ABCD. E, F are the mid. points of $\overline{AD}$, $\overline{BC}$ respectively. Prove that $\overline{AF}$ and $\overline{EC}$ trisect the diagonal $\overline{BD}$. (See problem 18.)

(17) Prove that if the angle bisectors of two consecutive angles of a $\parallel^{gm}$ have a point in common then they are perpendicular to each other.

(18) Prove that the lines through opposite vertices of a $\parallel^{gm}$ and mid. points of opposite sides trisect a diagonal.

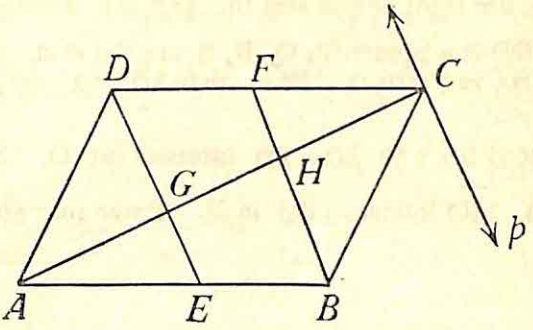


Fig. 165.

Given : A $\parallel^{gm}$ ABCD. E is the mid. point of $\overline{AB}$ and F is the mid point of $\overline{CD}$. $\overline{DE}$ intersects $\overline{AC}$ in G and $\overline{BF}$ intersects $\overline{AC}$ in H.
To prove that : (*i*) $\overline{AG} \cong \overline{GH}$ and (*ii*) $\overline{GH} \cong \overline{HC}$.

Proof. The ordering of points G, H on $\overline{AC}$ will be A–G–H–C as B, F lie on the oppoiste to A-side of $\overline{DE}$ and hence A–G–H. Similarly G–H–C as D, E lie on opposite to C-side of $\overline{FB}$.

Statement	*Reason*
(1) Through C, there exists a line *p*, parallel to $\overline{BF}$	(1) Parallel Postulate
(2) $\overline{EB} \cong \overline{DF}$	(2) Why ?
(3) $\overline{EB} \parallel \overline{DF}$	(3) Why ?
(4) DEBF is a $\parallel^{gm}$	(4) Statements 2 and 3 Theorem IX (18)
(5) $\overline{BF} \parallel \overline{DE}$	(5) Statement 4
(6) *p*, $\overline{BF}$, $\overline{DE}$ are parallel to each other	(6) Statements 1 and 5
(7) $\overline{CF} \cong \overline{FD}$	(7) Given
(8) $\overline{CH} \cong \overline{HG}$	(8) Statements 6 and 7 Theorem IX (20).

By considering a line through A, parallel to anyone of $\overleftrightarrow{DE}$, $\overleftrightarrow{FB}$ and by following similar reasoning as for $\overline{CH} \cong \overline{HG}$, we can prove that $\overline{HG} \cong \overline{GA}$. Hence we prove that the lines through opposite vertices of a $\|^{gm}$ and mid. points of opposite sides trisect a diagonal.

(19) E, F, G, H are mid. points of sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ of a quadrilateral ABCD respectively. Prove that EFGH is a $\|^{gm}$ and hence prove that $\overline{EG}$, $\overline{FH}$ bisect each other.

(20) If one angle of a $\|^{gm}$ is a right angle then all its remaining angles are right angles and the $\|^{gm}$ is a rectangle.

(21) ABCD is a square. P, Q, R, S are the mid. points of sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ respectively. Prove that $\overline{AQ}$, $\overline{AR}$, $\overline{CP}$, $\overline{CS}$ enclose a rhombus.

(22) ABCD is a $\|^{gm}$ $\overline{AC}$, $\overline{BD}$ intersect at O. $M \in \overline{AB}$ such that A—M—B. $\overrightarrow{MO}$ intersects $\overline{CD}$ in N. Prove that $\overline{MO} \cong \overline{ON}$.

CHAPTER X

LOCI

The word locus, the plural of which is 'loci' is the Latin word meaning 'place' or 'location'. These meanings of the word 'locus' donot at all explain the context in which it is used in geometry. In geometry, 'the locus of the points' is a geometric figure consisting of set of points and only those points that satisfy a given condition or set of conditions. A locus may consist of only a point or a line or several lines or a set of isolated points or a surface or a combination of these.

In a traditional course in geometry, the locus of a point is generally defined as a path traced out by a moving point under certain given conditions. The modern trend is to use 'the locus of *points*' rather than the 'locus of a *point*' and the concept of motion of a point is not very much accepted.

In fact with the coming into popular use of set terminology, there is not very much need to retain the term 'locus'. Mathematicians coined the word 'locus' to describe geometric problems involving sets of points. Now the term 'locus' is generally being replaced by the term 'set' and it may not be very long before the term locus gets out of use in a course of geometry and is completely replaced by the term set which has a universal applications in all Mathematics.

We may, however, define locus as a geometric figure which satisfies the given condition or set of conditions and it contains such points which do not satisfy the given condition. Hence locus is inclusive as well as exclusive. Therefore, while proving a theorem on locus, we have to show that the figure includes all those points that satisfy the given condition or set of conditions and at the same time excludes all those points that do not satisfy those conditions. There are two methods of attach for developing a proof of theorem on locus.

Method I. To prove that every point on the figure satisfies the given condition. Also to prove that every point that satisfies the given condition lies on the figure.

Method II. To prove that every point on the figure satisfies the given condition or set of conditions and also every point not on the figure does not satisfy the given condition or set of conditions.

We can make use of any one of these methods according to our convenience. It will be clear from what is stated above that in a theorem on locus, we have to prove that a statement p is equivalent to statement q or $p \leftrightarrow q$. In other words we have to prove that $p \rightarrow q$ and $q \rightarrow p$. So while proving a theorem on locus, we have to prove both the theorem and its converse. In fact a statement on locus is if and only if statement and should be proved as iff statement. In the language of sets, it can be said that for giving a complete proof of a theorem on locus, we have to prove that a geometrical figure consisting of set of points (Set A) is equivalent to set of points that satisfy the given condition or set of conditions (Set B). This implies that Set A = Set B or we have to prove that $A \subseteq B$ and $B \subseteq A$.

A statement involving the concept of 'locus' is :—

'In a plane, the locus of points equidistant from two given points is the perpendicular bisector of the line segments joining the given points. (Please note the use of the word 'points'.)

This statement is equivalent to following two statements :

(1) If a point lies on the perpendicular bisector of a line segment, it is equidistant from the ends of the line segment.

(2) If a point is equidistant from the ends of a line segment, it lies on the perpendicular bisector of that line segment.

If we prove the above theorem by proving these two statements, we will illustrate the first method of proving a theorem on locus.

Again this theorem is also equivalent to following two statements :

(1) If a point lies on the perpendicular bisector of a line segment, it is equidistant from the ends of the line segment.

(2) If a point does not lie on the perpendicular bisector of a line segment, it is not equidistant from the ends of the line segment.

We can prove this theorem by proving these two statements. This will illustrate our second method of proving a theorem on locus. We however intend to follow the first method as it is rather convenient. However, we can state this very theorem in language of sets. This is as follows.

Theorem X (1). The perpendicular bisector of a line segment in a plane is the set of all points of the plane that are equidistant from the end points of the line segment.

First we will prove that 'in a plane, every point on the perpendicular bisector of a line segment is equidistant from the end points of the line segment'.

Given : A line segment $\overline{AB}$ and a line $p \perp \overline{AB}$ at C where C is the mid. point of $\overline{AB}$ (or $\overline{AC} \cong \overline{CB}$). Also D is an arbitrary point lying on p.

To prove that : D is equidistant from A and B or $\overline{DA} \cong \overline{DB}$.

Proof. (1) If $D \in \overline{AB}$ (or D lies on $\overline{AB}$) then $D = C$. Hence D satisfies the given condition and there is nothing further to prove.

(2) If $D \notin \overline{AB}$, then :

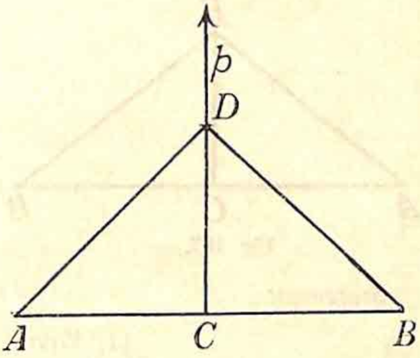


Fig. 166.

Statement	*Reason*
(1) $p \perp \overline{AB}$	(1) Given
(2) $\overline{DC} \perp \overline{AB}$	(2) Statement 1
(3) There is exactly one line containing A and D	(3) Postulate 2
(4) There is exactly one line containing D and B	(4) Postulate 2
(5) $m\angle DCA = m\angle DCB = 90$	(5) Statement 2
(6) $\overline{DC} = \overline{CD}$	(6) A line segment is congruent to itself
(7) $\overline{CA} \cong \overline{CB}$	(7) Given
(8) 1—1 correspondence $DCA \leftrightarrow DCB$ is a congruence and $\triangle DCA \cong \triangle DCB$	(8) S.A.S. Postulate
(9) $\overline{DA} \cong \overline{DB}$	(9) Statement 8
(10) D is equidistant from A and B	(10) Statement 9.

Now we have to prove that if a point is equidistant from the ends of a line segment, it lies on the perpendicular bisector of that line segment.

Given : A line segment $\overline{AB}$ and C is the mid. point of $\overline{AB}$. E is a point such that $\overline{EA} \cong \overline{EB}$.

To prove that : E lies on the perpendicular bisector of $\overline{AB}$.

Proof. (1) If $E \in \overline{AB}$, then $E = C$ and we have nothing further to prove as we have already proved that at a given point on a given line in a given plane, there exists one and only one perpendicular to the line.

(2) If $E \notin \overline{AB}$ then :

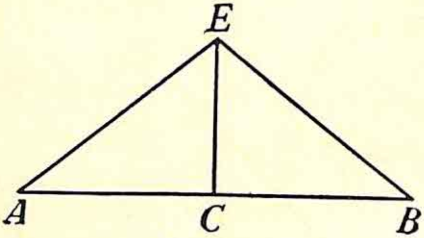


Fig. 167.

Statement	*Reason*
(1) $\overline{EA} \cong \angle \overline{EB}$	(1) Given
(2) $\angle EAC \cong \angle EBC$	(2) EAB is isosceles $\triangle$
(3) $\overline{AC} \cong \overline{CB}$	(3) Given
(4) 1—1 correspondence $ECA \leftrightarrow ECB$ is a congruence and $\triangle ECA \cong \triangle ECB$	(4) S.A.S. Postulate
(5) $\angle ECA \cong \angle ECB$	(5) Corresponding parts of congruent triangles
(6) $\angle ECA$, $\angle ECB$ form a linear pair	(6) $\overrightarrow{CA}$ is opposite to $\overrightarrow{CB}$
(7) $m\angle ECA = m\angle ECB = 90$	(7) Statements 5 and 6
(8) E lies on perpendicular bisector of $\overline{AB}$	(8) Statement 7.

Theorem X (2). The right bisectors (perpendicular bisectors) of the sides of a triangle are concurrent.

Given : $\triangle ABC$. D, E, F are the mid. points of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively.

To prove that : The perpendiculars to the sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ at the points D, E and F respectively are concurrent (meet at a point.)

Proof.

Statement	*Reason*
(1) $\overrightarrow{DK} \perp \overline{AB}$ and $\overrightarrow{FM}$ is perpendicular to $\overline{AC}$; $\overline{DK} \cap \overrightarrow{FM} = O$	(1) $\overrightarrow{DK}$ and $\overrightarrow{FM}$ are not parallel to each other (why ?) ; hence meet at a point
(2) $\overline{OA} \cong \overline{OB}$	(2) Theorem X (1) $O \in \overrightarrow{DK}$

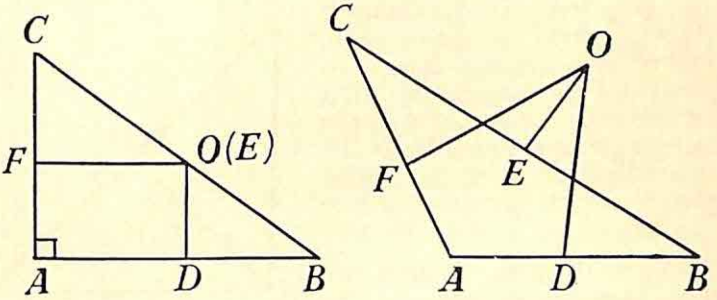


Fig. 168.

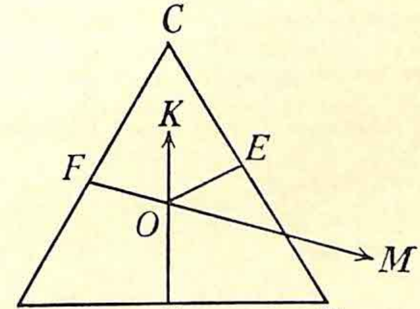


Fig. 169.

Statement	*Reason*
(3) $\overline{OA} \cong \overline{OC}$	(3) Theorem X (1); $O \in \overrightarrow{FM}$
(4) $\overline{OB} \cong \overline{OC}$	(4) Statements 2 and 3
(5) O lies on perpendicular bisector of $\overline{BC}$	(5) Statement 4 Theorem X (1)
(6) O lies on the perpendicular bisectors of $\overline{AB}$, $\overline{AC}$, $\overline{BC}$	(6) Statements 1 and 5
(7) The perpendicular bisectors of sides of a triangle are concurrent	(7) Statement 6.

The point of concurrency of the perpendicular bisectors of sides of a triangle is called circumcentre.

Theorem X (3). In a triangle, the altitudes are concurrent.

Given : $\triangle ABC$. $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are the attitudes respectively to the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$.

To prove that : $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are concurrent.

Proof. Now either the triangle ABC is right angle, obtuse angle or acute angle. We have already proved that [Problem D (8) Chapter IX] if $\angle C$ of $\triangle ABC$ is obtuse angle then the foot of perpendicular from A to $\overline{BC}$ is not a point of segment $\overline{BC}$. On similar lines we can prove that if all the angles of a triangle are acute then the foot of perpendicular from any vertex to the opposite side is a point of that side.

Now in case the given triangle is a right triangle, then the altitudes are concurrent. Let ABC be the right triangle with $m\angle B=90$. Now $\overline{AB}$ is the altitude to $\overline{BC}$ and $\overline{BC}$ is the altitude to $\overline{AB}$ with B being the common point between the altitudes. Altitude $\overline{BE}$ to $\overline{AC}$ also contains B. Hence in this case the altitudes are concurrent.

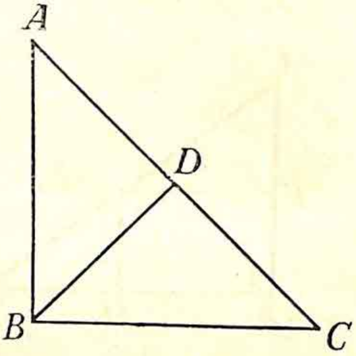


Fig. 170.

The proof in the case when the triangle is acute is as under :—

Proof.

Statement	*Reason*
(1) Through A, there exists a line $\overleftrightarrow{XY} \parallel \overline{BC}$; Through B, there exists a line $\overleftrightarrow{MN} \parallel \overline{AC}$ and through C, there exists a line $\overleftrightarrow{PQ} \parallel \overline{AB}$	(1) Parallel Postulate
(2) $\overleftrightarrow{XY}$ and $\overleftrightarrow{PQ}$ have a common point H	(2) $\overleftrightarrow{XY} \nparallel \overleftrightarrow{PQ}$ as $\overline{BC} \nparallel \overline{AB}$
(3) H is on C-side of $\overline{AB}$ and A-side of $\overline{BC}$	(3) As $\overleftrightarrow{PQ} \parallel$ AB and $\overleftrightarrow{XY} \parallel$ BC

(4) ABCH is a convex quadrilateral	(4) Statement 3

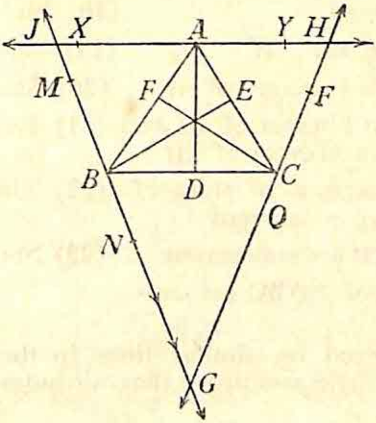


Fig. 171.

(5) $\overline{AB} \parallel \overline{CH}$ and $\overline{BC} \parallel \overline{AH}$	(5) Statement 1
(6) ABCH is a $\parallel^{gm}$	(6) Statements 4 and 5
(7) Also ACBJ is a $\parallel^{gm}$ where $\overleftrightarrow{MN} \cap \overleftrightarrow{XY} = J$ and J lies on A-side of $\overline{BC}$ and B-side of $\overline{AC}$	(7) Reasoning on similar lines as for $\parallel^{gm}$ ABCH
(8) $\overline{BC} \cong \overline{AH}$	(8) Statement 6
(9) $\overline{BC} \cong \overline{AJ}$	(9) Statement 7
(10) $\overline{AH} \cong \overline{AJ}$	(10) Statements 8 and 9
(11) J—A—H	(11) Statements 3 and 7
(12) A is mid. point of $\overline{JH}$	(12) Statements 10 and 11
(13) B is mid. point of $\overline{JG}$ where $\overleftrightarrow{MN} \cap \overleftrightarrow{PQ} = G$ and C is mid. point of $\overline{GH}$	(13) Reasoning similar as for statement 12
(14) $m\angle DAB = 90 - m\angle B$	(14) $m\angle ADB = 90$; also angle-sum property of a triangle
(15) $m\angle JAB = m\angle B$	(15) Alternate interior angles
(16) $m\angle DAB + m\angle JAB = 90$	(16) Statements 14 and 15

(17) $\overline{AB}$ is between $\overline{AD}$ and $\overline{AJ}$	(17) $\overline{AB}$ is diagonal of ▯^{gm} AJBC
(18) $m\angle JAD=90$	(18) Statement 16 and 17
(19) $\overline{AD}\perp\overline{AJ}$ or $\overline{AD}\perp\overline{JH}$	(19) Statement 18
(20) $\overline{AD}$ is right bisector of $\overline{JH}$	(20) Statements 12 and 19
(21) $\overline{BE}$ is right bisector of $\overline{JG}$ and $\overline{CF}$ is right bisector of $\overline{GH}$	(21) Reasoning similar as for $\overline{AD}$
(22) Right bisectors of sides of $\triangle JGH$ are concurrent	(22) Theorem X (2)
(23) $\overline{AD}$, $\overline{BE}$, CF are concurrent	(23) Statements 20, 21, 22
(24) Altitudes of $\triangle ABC$ are concurrent	

We can proceed on similar lines in the case the triangle is having one obtuse angle and prove that altitudes of the triangle are concurrent.

The point of concurrency of the altitudes is called the orthocentre.

Theorem X (4). The bisectors of the angles between two intersecting lines, excluding the point of intersection, is the set of all points that are equidistant from two lines.

This statement is in set terminology and is equivalent to the following statement involving the use of word locus :

"The locus of a point which is equidistant from two given intersecting lines, is the pair of bisectors of the angles between the lines".

In fact a better statement of this theorem using the word locus will be :

"In a plane, the locus of points which are in the interior of an angle and which are equidistant from its sides is the bisector of the angle."

To prove this theorem, we are to prove that : (1) If a point lies on the bisector of an angle then it is equidistant from the sides of the angle. (2) Any point equidistant from the sides of the angle lies on the bisector of the angle.

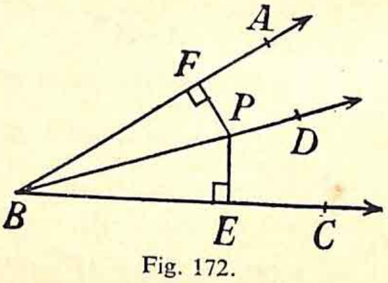


Fig. 172.

Given : $\overrightarrow{BD}$ the angle bisector of $\angle ABC$ and $P\in\overrightarrow{BD}$ (P is a point lying on $\overrightarrow{BD}$). $\overline{PE}\perp\overline{BC}$ and $\overline{PF}\perp\overline{AB}$.

To prove that : $\overline{PE}\cong\overline{PF}$.

Proof.

Statement	*Reason*
(1) $\angle PBE \cong \angle PBF$	(1) Given
(2) $\angle PEB \cong \angle PFB$	(2) Measure of each is 90
(3) $\overline{PB} \cong \overline{BP}$	(3) Any segment is congruent to itself
(4) 1—1 Correspondence $PBE \leftrightarrow PBF$ is a congruence and $\triangle PBE \cong \triangle PBF$	(4) S.A.A. Theorem
(5) $\overline{PE} \cong \overline{PF}$	(5) Corresponding points of congruent triangles.

Now we prove the second statement :

Given : An $\angle ABC$ and P any point in the interior of $\angle ABC$ such that $\overline{PE} \cong \overline{PF}$ where $\overline{PE} \perp \overrightarrow{BC}$ and $\overline{PF} \perp \overrightarrow{BA}$.

To prove that : P lies on the angle bisector of $\angle ABC$.

Proof.

Statement	*Reason*
(1) $\overline{PE} \cong \overline{PF}$	(1) Given
(2) $\overline{PB} \cong \overline{BP}$	(2) Any segment is congruent to itself
(3) Measure of each of $\angle PEB$ and $\angle PFB$ is 90	(3) Given
(4) 1—1 correspondence $PEB \leftrightarrow PFB$ is a congruence and $\triangle PEB \cong \triangle PFB$	(4) Hypotenuse-leg Theorem

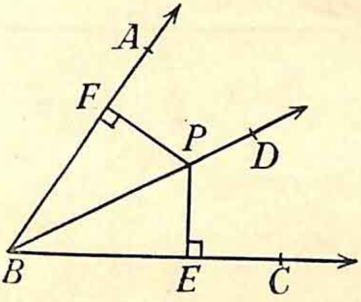


Fig. 173.

(5) $\angle PBE \cong \angle PBF$	(5) Corresponding parts of congruent triangles
(6) P lies on the angle bisector of $\angle ABC$	(6) Statement 5

Now we are in a position to prove that the bisectors of the angles between two intersecting lines excluding the point of intersection is the set of all points that are equidistant from the lines. If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two lines intersecting at O, we can consider the angle bisectors of $\angle BOD$, $\angle BOC$, $\angle COA$, $\angle DOA$ and prove that the angle bisectors represent the required locus.

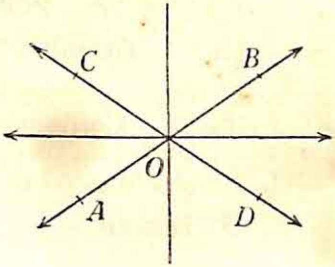


Fig. 174.

Theorem X (5). In any triangle, the bisectors of angles are concurrent.

Given : A $\triangle ABC$. $\overrightarrow{AX}$, $\overrightarrow{BY}$, $\overrightarrow{CZ}$ are the angle bisectors of the angles $\angle A$, $\angle B$ and $\angle C$ respectively.

To prove that : $\overrightarrow{AX}$, $\overrightarrow{BY}$ and $\overrightarrow{CZ}$ and concurrent.

Proof.

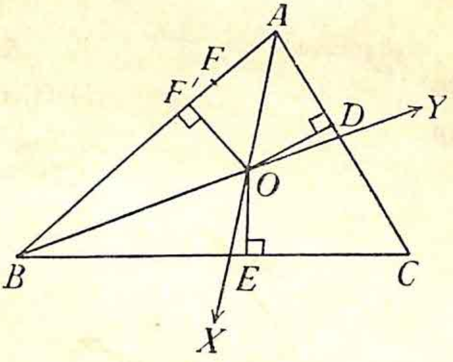


Fig. 175.

Statement	*Reason*
(1) $\overrightarrow{AX}$, $\overrightarrow{BY}$ have a common point O in the interior of $\triangle ABC$	(1) $\overrightarrow{AX}$ lies in the interior of $\angle CAB$ and $\overrightarrow{BY}$ lies in the interior of $\angle ABC$. Hence $\overrightarrow{AX} \cap \overrightarrow{BY} \in \angle CAB \cap \angle ABC$ or O lies in the interior of $\triangle ABC$

(2) $\overline{OE} \perp \overline{BC}$; $\overline{OF} \perp \overline{AB}$ and $\overline{OD} \perp \overline{AC}$ exists uniquely	(2) There exists a unique perpendicular from any point in a plane to a line segment lying in the same plane (the given point not lying in the given plane)
(3) $\overline{OE} \cong \overline{OF}$	(3) Theorem X (4)
(4) $\overline{OF} \cong \overline{OD}$	(4) Theorem X (4)
(5) $\overline{OE} \cong \overline{OD}$	(5) Statements 3 and 4
(6) O lies on the angle bisector of $\angle ACB$	(6) Statement 5
(7) $O \in \overrightarrow{CZ}$	(7) Statement 6
(8) $\overrightarrow{AX}$, $\overrightarrow{BX}$, $\overrightarrow{CZ}$ have a common point O	(8) Statements 1 and 7
(9) In a triangle, the bisectors of the angles are concurrent.	(9) Statement 8.

The point of concurrence of the bisectors of angles of a triangle is called in-centre.

Theorem X (6). In a triangle, the medians are concurrent and the point of concurrency is the point of trisection of each median.

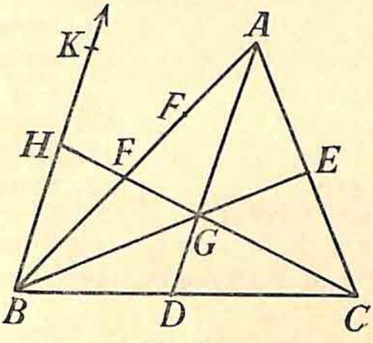


Fig. 176.

Given : $\triangle$ABC. D, E, F are the mid points of sides $\overline{BC}$, CA and AB respectively.

To prove that : $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are concurrent.

Proof.

Statement	*Reason*
(1) $\overline{AD}$ and $\overline{BE}$ have a common point G which lies in the interior of $\triangle ABC$.	(1) $\overline{AD}$ lies in the interior of $\angle CAB$ and $\overline{BE}$ lies in the interior of $\angle ABC$. Hence $\overline{AD} \cap \overline{BE} \in \angle CAB \cap \angle ABC$ or G lies in the interior of $\triangle ABC$
(2) $\overrightarrow{CG} \cap \overline{AB} = F'$ such that $A-F'-B$ and $F' \neq F$	(2) Cor. to Pasch's theorem
(3) $\overrightarrow{BK} \parallel \overline{DG}$ exists	(3) Parallel Postulate
(4) $\overrightarrow{CF'}$, $\overrightarrow{BK}$ have a common point H	(4) As $\overrightarrow{CG}$ have a common point with $\overline{DG}$, it must have a common point with $\overrightarrow{BK}$ as $\overrightarrow{BK} \parallel \overline{DG}$
(5) In $\triangle CBH$, $\overline{DG} \parallel \overline{BH}$ and D is mid. point of $\overline{BC}$	(5) Statement 3. Given
(6) G is mid. point of $\overline{CH}$	(6) Statement. Result of problem 11 of problem set of Chapter IX
(7) $\overline{CG} \cong \overline{GH}$	(7) Statement 6
(8) In $\triangle CAH$, G is mid. point of $\overline{CH}$ and E is mid. point of $\overline{CA}$	(8) Statement 6. Given
(9) $\overline{GE} \parallel \overline{AH}$	(9) Statement 8. Result of problem 11 of Chapter IX
(10) $\overline{BG} \parallel \overline{AH}$	(10) Statement 9
(11) $\overline{BH} \parallel \overline{AG}$	(11) Statement 3
(12) BGAH is a $\parallel^{gm}$	(12) Statements 10 and 11
(13) $\overline{BF'} \cong \overline{AF'}$ and $\overline{GF'} \cong \overline{F'H}$	(13) Diagonals of $\parallel^{gm}$. bisect each other
(14) F′ is mid. point of $\overline{AB}$ and hence $F' = F$	(14) Statement 13, given. A line segment cannot have more than one mid. point
(15) $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ have a common point G and hence these are concurrent	(15) Statements 1, 2, 14

(16) $m(\overline{GF})=\frac{1}{2}\,m(\overline{CG})$	(16) Statements 7 and 13
(17) G is the point of trisection of $\overline{CF}$	(17) Statement 16
(18) G is point of trisection of $\overline{AD}$ and $\overline{BE}$.	(18) Reasoning similar as for 17

G, the point of intersection of the medians of a $\triangle$ is known as centroid.

Problem Set

(1) Prove that the perpendicular bisectors of the sides of a triangle meet in a point, which is equidistant from the vertices.

(2) Given a $\triangle ABC$. The bisectors of $\angle A$ and the bisectors of exterior angles at B and C meet in a point which is equidistant from the sides.

(3) Given a $\triangle ABC$. Prove that the bisectors of $\angle A$ and $\angle B$ intersect in a point in the interior of the triangle. Then prove that the bisectors of angles of a triangle are concurrent.

(4) Given a $\triangle ABC$. D, E are points on the rays opposite to $\overrightarrow{BA}$ and $\overrightarrow{CA}$. Prove that the bisectors of $\angle CBD$ and $\angle BCE$ intersect in the interior of $\angle A$. Hence prove that the bisectors of any two exterior angles of a triangle are concurrent with the bisector of the interior angle at the remaining vertex, at a point which lies in the interior of that angle.

(5) Two isosceles triangles stand on a common base ; prove that the line segment joining their vertices bisects the base at right angles.

(6) Prove that the bisectors of exterior angles of a triangle enclose a triangle whose vertices are equidistant from the sides of the given triangle.

(7) Show that there are altogether four points equidistant from the sides of a triangle.

(8) Given a point P in the plane of $\overleftrightarrow{AB}$, $P \notin \overleftrightarrow{AB}$, A_1, A_2, A_3...are points on $\overleftrightarrow{AB}$. What will be the locus of mid. points of line segments $\overline{PA}$, $\overline{PA_1}$, $\overline{PA_2}$, $\overline{PA_3}$...$\overline{PB}$?

(9) If O is the orthocentre of $\triangle ABC$, then prove that B is the orthocentre of $\triangle AOC$ and A is the orthocentre of $\triangle BOC$.

(10) If O is the orthocentre of $\triangle ABC$ and D, E, F are the mid points of $\overline{AO}$, $\overline{BO}$ and $\overline{CO}$, prove that O is also the orthocentre of $\triangle DEF$.

CHAPTER XI

SPACE GEOMETRY

The geometrical figures that we have been discussing so far are called plane figures since the set of points (points and lines) constituting these figures lie in the same plane. In space geometry, figures also consist of points and lines but all of these will not lie in the same plane. Plane figures have either one or two dimensions. A line has only one dimension—its length. A rectangle has two dimensions—length and breadth. Figures in space have either one or two or three dimensions. A cuboid is an example of a figure in space geometry and it has three dimensions—length, breadth and height.

We have already discussed postulates about the plane and the space and it may not be out of place to mention those here again :—

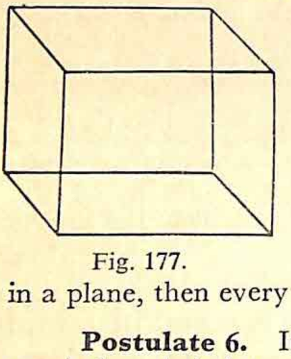


Fig. 177.

Postulate 3. A plane is a set of points and contains atleast three non-collinear points.

Postulate 4. If A, B, C are any three non-collinear points, then there is one and only one plane that contains all of them.

Postulate 5. If two points of a line lie in a plane, then every point of that line lies in that plane.

Postulate 6. If two planes intersect (have a point in common) then their intersection is a line.

Postulate 7. Space contains atleast four points which are non-collinear and non-coplanar.

We have also proved the following theorems about planes and space in the chapter on 'Incidence postulates and theorems' :—

Theorem. The intersection of a line that does not lie in a plane but intersects it, is only one point.

Theorem. Given a line and a point not on the line, then there is one and only one plane that contains both the line and the plane.

Theorem. If two planes intersect, their intersection is only one line.

Theorem. Two intersecting lines determine exactly one point.

Theorem. Space contains atleast six lines.

Theorem. Space contains atleast four planes.

Although a plane is indefinite in extent, we represent it as a $\|^{gm}$. or as a rectangle seen obliquely, we call it by naming one of its diagonals as plane AC or giving it a name like π etc.

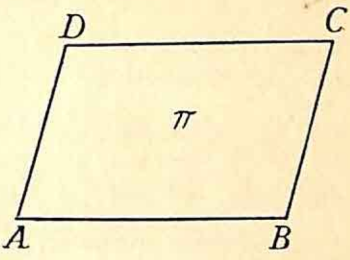


Fig. 178.

Perpendiculars and Parallels in space

Perpendicularity of a line and a plane. When are a line and a plane not containing that line perpendicular to each other ? A line that is not in the given plane but intersects it and is perpendicular to every line lying in the given plane and passing through the point of intersection (We know that intersection is only one point) of the line and the plane, is said to be perpendicular to the plane. If a line is perpendicular to the plane, the plane is also said to be perpendicular to the line. We can state the following definition in this regard.

Definition XI (1). A line and a plane not containing the line are said to be perpendicular to each other if these intersect and if every line that passes through their point of intersection and lies in the given plane is perpendicular to the given line.

Sometimes it is very hard to visualise three dimensional figures as we represent them on two dimensional plane of the paper. For instance to represent a line perpendicular to the plane we make use of the following two dimensional drawing. We read this drawing as follows :

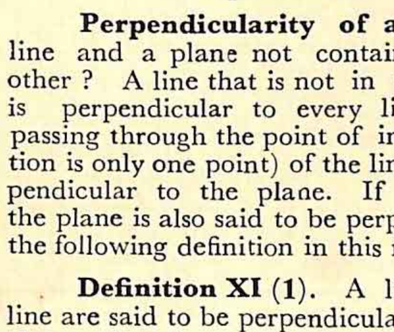


Fig. 179.

A line $\overleftrightarrow{AP}$ intersects the plane π at P and as we are given that this line is perpendicular to the plane the measure of each of the angles $\angle APB$, $\angle APC$, $\angle APE$, $\angle APD$ is 90, although the figure apparently does not show it. If you feel any difficulty, then you can draw the plane and the rays $\overrightarrow{PB}$, $\overrightarrow{PC}$, $\overrightarrow{PE}$, $\overrightarrow{PD}$ etc. and then you can represent the line perpendicular to the plane by a match stick and then you will see that the measure of each of these angles is 90.

In this figure, we have indicated the lines $\overleftrightarrow{BP}$, $\overleftrightarrow{DP}$, $\overleftrightarrow{EP}$ and $\overleftrightarrow{CP}$

lying in the plane π which are perpendicular to the line $\overleftrightarrow{AP}$. In fact at any point P on the line $\overleftrightarrow{AP}$, there are many different lines which are coplanar with each other and which are each perpendicular to $\overleftrightarrow{AP}$. Hence we say that the line $\overleftrightarrow{AP}$ is $\perp$ to the plane π which contains all these lines that are perpendicular to the given line at the given point P. In fact if a plane π contains atleast two lines which are perpendicular to the given line $\overleftrightarrow{AP}$, then $\pi \perp \overleftrightarrow{AP}$ or $\overleftrightarrow{AP} \perp \pi$, but we need to prove this :

Theorem XI (1). If a line is perpendicular to each of the two intersecting lines at their point of intersection then it is perpendicular to every line which passes their point of intersection and which lies in their plane (or then it is perpendicular to the plane that contains these lines).

Given : Two lines, l_1 and l_2, intersecting at a point A and lying in a plane π (π is simply the name of the plane) A line l is perpendicular to each of the lines l_1 and l_2 at A (l will not be coplanar with l_1 and l_2 as then it cannot be perpendicular to both l_1 and l_2).

To prove that : If l_3 is any other line lying in the plane π and passing through A, then $l \perp l_3$ and hence every line through A and lying in π is perpendicular to l or l is perpendicular to every line passing through A and lying in the plane π.

Proof. If $l_3 = l_1$ or $l_3 = l_2$ then we have nothing further to prove. So let $l_3 \neq l_1$ and also $l_3 \neq l_2$. Also on l_1, l_2, l_3, there are rays r_1, r_2, r_3 having a common end point A such that r_3 lies between r_1 and r_2.

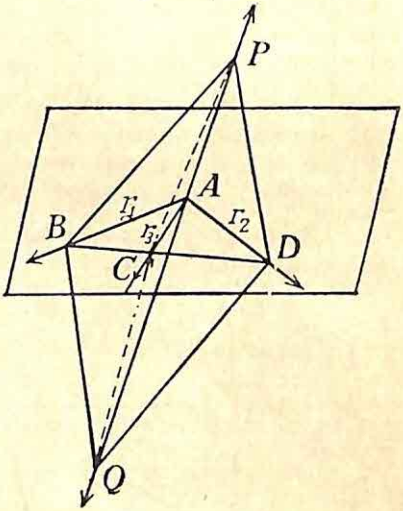


Fig. 180.

Statement	*Reason*
(1) There exists a point B on r_1 and D on r_2 such that $\overline{AB} \cong \overline{AD}$	(1) Segment construction theorem
(2) $\overline{BD}$ and r_3 have a common point C such that B–C–D	(2) $\overline{BD}$ and r_3 are coplanar and r_3 lies between r_1 and r_2. (Also cor. to Pasch's Postulate).

(3) There exists a point Q on the ray opposite to $\overrightarrow{AP}$ such that $\overline{AP} \cong \overline{AQ}$	(3) Point plotting theorem
(4) $\angle PAB \cong \angle PAD$ and $\angle QAB \cong \angle QAD$	(4) Measure of each being 90
(5) $1-1$ correspondences $PAB \leftrightarrow PAD \leftrightarrow QAB \leftrightarrow QAD$ are congruences and $\triangle PAB \cong \triangle PAD \cong \triangle QAB \cong \triangle QAD$	(5) S.A.S. Postulate (How $\triangle PAB \cong \triangle QAB$ etc ?)
(6) $\overline{PB} \cong \overline{PD} \cong \overline{QB} \cong \overline{QD}$	(6) Corresponding parts of congruent triangles
(7) $1-1$ correspondence $PBD \leftrightarrow QBD$ and $\triangle PBD \cong QBD$	(7) S.S.S. Theorem. How ?
(8) $\angle PBD \cong \angle QBD$	(8) Corresponding parts of congruent triangles
(9) $1-1$ correspondence $PBC \leftrightarrow QBC$ is a congruence and $\triangle PBC \cong \triangle QBC$	(9) S.A.S. Postulate. How ?
(10) $\overline{PC} \cong \overline{QC}$	(10) Corresponding parts of congruent triangles
(11) $1-1$ correspondence $PAC \leftrightarrow QAC$ is a congruence and $\triangle PAC \cong \triangle QAC$	(11) S.S.S. Theorem. How ?
(12) $\angle PAC \cong \angle QAC$	(12) Corresponding parts of congruent triangles
(13) $\angle PAC$, $\angle QAC$ form a linear pair	(13) $\overrightarrow{AP}$ and $\overrightarrow{AQ}$ are opposite rays
(14) Measure of each of $\angle PAC$, $\angle QAC$ is 90	(14) Statements 12 and 13.

Theorem XI (2). [*Converse to theorem XI (1).*] All lines which are perpendicular to a given line at a given point lie in the same plane.

Given : A line l and two lines l_1 and l_2 each perpendicular to l at the point A. The plane π contains both l_1 and l_2. Another line l_3 is perpendicular to l at A.

To prove that : l_3 lies in π.

Proof. There are two possible alternatives :

(*i*) l_3 lies in π. In this case there is nothing further to prove.

(*ii*) l_3 does not lie in π ($l_3 \notin \pi$). Then :

Statement	*Reason*
(1) l_3 and l determine a plane p	(1) Theorem : Two intersecting lines determine exactly one plane
(2) The planes p and π intersect	(2) They have a point A in common
(3) Planes p and π have a line l_4 (say) in common	(3) Postulate 6
(4) l_4 is $\perp$ l at A	(4) As l_4 lies in π Theorem XI (1)
(5) l, l_3, l_4 are coplanar	(5) l_4 lies in p the plane determined by l and l_3
(6) $l_4 \perp l$ and $l_3 \perp l$	(6) Statement 4 and given
(7) Two lines l_4, l_3 lie in the plane of another line l and are perpendicular to l at A.	(7) Statement 6.

Statement **7** contradicts the theorem VI (3) which we have already proved. Hence the assumption that $l_3 \notin \pi$ has to be abandoned. Theorem XI (1) and XI (2) together establish the existence and uniqueness of a plane perpendicular to a given line at a given point on the given line which we can state as follows :

'At each point of a given line, there is a unique plane containing the given point and perpendicular to the line at that point'. Another statement would be : 'Through a given point on a given line, there passes one and only one plane perpendicular to the line'. We can however prove its following generalised form.

Theorem XI (3). There exists a unique plane containing a given point and perpendicular to a given line. (The given point may or may not lie on the given line).

Given : A line l and a point A.

To prove that : (1) There exists at least one plane π, through A and perpendicular to l.

(2) There exists atmost (only one) one plane through A and perpendicular to l.

Proof. There are two possible alternatives.

(i) A lies on l. In this case we have nothing further to prove as theorems XI (1) and XI (2) establish the existence as well as uniqueness of a plane perpendicular to a given line at a given point on it.

(ii) A does not belong to l ($A \notin l$). Then :

Statement	*Reason*
(1) There exists a unique line l_1 containing A and perpendicular to line l.	(1) Theorem VII (6)
(2) l and l_1 have a common point P ($l \cap l_1 = P$)	(2) Theorem : If two lines intersect then their intersection is atmost a point.
(3) There exists a unique plane $\pi \perp$ to l at P	(3) Theorem XI (1) and XI (2)
(4) l_1 lies in π	(4) Since π contains all lines $\perp$ to l at P. [Theorem XI (2)]
(5) π contains A and is $\perp$ to l	(5) Statements 1, 3 & 4.

So we establish that there exists atleast one plane through A and $\perp$ to l. Now we want to prove that there exists atmost one plane through A and $\perp l$. Let us assume the contrary and state that there exists another plane p, through A and perpendicular to l. Then :

(6) The line l and the plane p have a point B in common (intersect at B)	(6) Def. of perpendicularity of a line and a plane.
(7) $B \neq P$	(7) $B = P$ leads to a contradiction of theorems XI (1) and XI (2) as two planes cannot be perpendicular to a given line at a given point.
(8) $\overleftrightarrow{AB}$ is $\perp l$.	(8) $\overleftrightarrow{AB}$ lies in plane p, which we have assumed perpendicular to l[Theorem XI (1)]
(9) $\overleftrightarrow{AP}$ (l_1) is $\perp l$	(9) Statement 1

(10) Two distinct lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AP}$ having a common point A are perpendicular to l.	(10) Statements 7, 8 & 9.

Statement 10 contradicts the theorem VII (6) which we have already proved. Therefore the assumption that there exists another plane p through A and perpendicular to l need to be abandoned.

Hence there exists one and only one plane through A perpendicular to l.

Theorem XI (4) : Through a given point in a given plane, there exists one and only one line perpendicular to the plane.

Given : A plane π and a point $A \in \pi$.

To prove that : (1) There exists at least one line through A and perpendicular to π.

(2) There exists atmost one line through A and perpendicular to π.

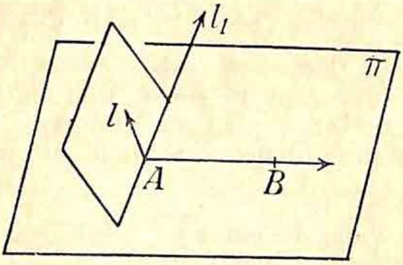


Fig. 181.

Proof :

Statement	*Reason*
(1) There exists a point $B \in \pi$	(1) Postulate 3
(2) At A, there exists a plane P—perpendicular to $\overleftrightarrow{AB}$	(2) Theorems XI (1) and XI (2)
(3) Plane π and plane P, have a common point A	(3) Statement 2
(4) π and P intersect in a line l_1 (say)	(4) Postulate 6.

(5) There exists a line $l \in P$ such that $l \perp l_1$ at A	(5) At each point of a line there exists a line which is perpendicular to the given line and lies in the plane containing the line. Theorem VI (3).
(6) $l \perp \overleftrightarrow{AB}$	(6) Def. of perpendicularity of a line and a plane.
(7) $l \perp l_1$	(7) Statement 5.
(8) l is perpendicular to the plane determined by $\overleftrightarrow{AB}$ and l_1	(8) Statements 6 and 7 Theorem XI (1)
(9) l is perpendicular to π	(9) Statement 8. Also as two lines l_1 and $\overleftrightarrow{AB}$ both lying in π determine a unique plane
(10) l is perpendicular to π at A	(10) Statements 5, 7 and 9.

So we have proved that there exists atleast one line through A and perpendicular to π. Now we want to prove that there exists atmost one line through A perpendicular to π. We assume the contrary and state that two distinct lines l and l_2 are each perpendicular to π at A. Then :

Statement	*Reason*
(1) There exists a plane Q determined by l and l_2	(1) Two intersecting lines determine exactly one plane (Theorem)
(2) Q and π have a point A in common	(2) Assumption
(3) Q and π intersect in a line l_3.	(3) Postulate 6
(4) l and l_2 are each perpendicular to l_3 at A	(4) Theorem XI (1)
(5) l_3 lies in both Q and π	(5) Statement 3
(6) In plane Q, two lines l and l_2 are perpendicular to a coplanar line l_3 at A.	(6) Statements 4 and 5

The statement 6 contradicts the theorem VI (3) which we have already proved. Hence the assumption that two distinct lines l and l_2 are each perpendicular to plane π at A has to be abandoned.

This leads us to assert that there exists one and only one line perpendicular to a given plane at a given point in it.

Theorem XI (5). Through a given point not in a given plane, there exists one and only one line perpendicular to the given plane.

Given : A plane π and a point $A \notin \pi$.

To prove that : (1) Through A, there exists atleast one line perpendicular to π.

(2) Through A, there exists atmost one line perpendicular to π.

Proof.

Statement	*Reason*
(1) There exists a point $P \in \pi$	(1) Postulate 3
(2) l, a unique line, perpendicular to π at P exists.	(2) Theorem XI (4)

Now if l contains A, then there is nothing further to prove. Let us assume that l does not contain A. Then

(3) l and A, determine a plane E	(3) Theorem : given a line and a point not on the line, then there is one and only one plane that contains both the line and the point

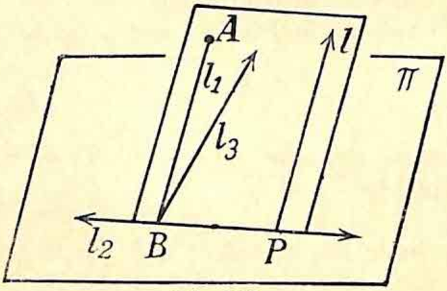


Fig. 182.

(4) π and E intersect in a line l_2	(4) Postulate 6
(5) From the point A, a unique line $l_1 \perp l_2$ at the point B exists	(5) Theorem VII (6)
(6) l_3, a unique line perpendicular to π at B exists	(6) Theorem XI (4)

If $l_3 = l_1$ then there is nothing further to prove, as it will imply the existence of a line through A perpendicular to π. Assume $l_3 \neq l_1$.

(7) $l_3 \perp l_2$	(7) As l_3 is perpendicular to π at B and l_2 is any line of π through B (Definition of perpendicularity of a line and a plane)
(8) Two distinct lines l_1 and l_3 are perpendicular to l_2 at B.	(8) Statements 5 and 7.

Statement 8 contradicts the theorem VI (3) that we have already proved. Hence the assumption that $l_3 \neq l_1$ has to be abandoned. This leads us to assert that through A, there exists atleast one line perpendicular to π. Now we want to prove that there exists only one line through A perpendicular to π. Let us assume the contrary and assert that through A, there exist two lines l_1 and l_3 both perpendicular to the plane π. Let E be the plane determined by l_1 and l_3. Planes E and π intersect in a line l_2 (say). (Postulate 6). l_1 and l_3 will both be perpendicular to l_2. (Definition of perpendicularity of a line and a plane). Let l be the perpendicular from A to l_2. (Theorem VII (6). If $l = l_1$ as well as $l = l_3$ then there is nothing further to prove as it implies, $l_2 = l_3 = l$. Assume $l \neq l_1$ and $l \neq l_3$. Then we have three lines l, l_1 and l_3 all containing the point A and perpendicular to the same line l_2. This contradicts the theorem VII (6). Hence the assumptions $l_1 \neq l_3$ or $l \neq l_1$ or $l \neq l_2$ need to be abandoned. Of course the following alternatives which are also contained in the preceding statement *viz.*, (1) $l = l_1$ but $l \neq l_3$ and (2) $l = l_3$ but $l \neq l_1$ also leads to a contradiction of theorem VII (6). Hence $l = l_1$ and $l = l_3$. This leads us to assert that there exists one and only one line through A, perpendicular to π.

Definition XI (2). (*Oblique line*). A line which intersects a plane exactly in one point but is not perpendicular to the plane is said to be oblique line.

Theorem XI (6) If from a point on a perpendicular to a plane, unequal line segments intersect the plane, the respective line segments from their point of intersection to the point of intersection of perpendicular and the plane are unequal in the same order.

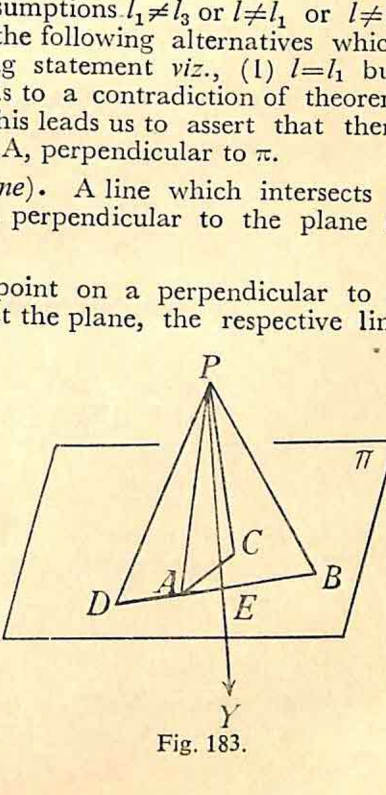


Fig. 183.

Given : A plane π and a line $l \perp \pi$. P is a point on l. $\overleftrightarrow{PB}$, $\overleftrightarrow{PC}$ are oblique lines to the plane π, B and C belong to π. Also $m(\overline{PB}) > m(\overline{PC})$. A is the foot of perpendicular from P to π.

To prove that : $m\overline{AB} > m\overline{AC}$

Proof.

Statement	*Reason*
(1) D is a point on the ray opposite to $\overrightarrow{AB}$ such that $\overline{AD} \cong \overline{AC}$	(1) Point plotting theorem
(2) $\angle PAD \cong \angle PAC$	(2) Measure of each is 90
(3) $\overline{PA} \cong \overline{PA}$	(3) Any segment is congruent to itself.
(4) 1−1 correspondence PAC↔PAD is a congruence and $\triangle PAC \cong \triangle PAD$	(4) S.A.S. Postulate
(5) $\overline{PC} \cong \overline{PD}$	(5) Corresponding parts of congruent triangles
(6) PBD is a $\triangle$	(6) $\overrightarrow{AD}$ is opposite to $\overrightarrow{AB}$ (statement 1)
(7) $m(\overline{PB}) > m(\overline{PD})$	(7) $m(\overline{PB} > m(\overline{PC})$ given and $\overline{PC} \cong \overline{PD}$. Statement 5
(8) $m\angle PDA > m\angle PBA$	(8) Theorem VIII (3)
(9) $m\angle PDA + m\angle PAD + m\angle DPA = 180$	(9) Angle-sum property of a $\triangle$
(10) $m\angle PDA + m\angle DPA = 90$	(10) Statement 9 : $m\angle PAD = 90$
(11) $m\angle PBA + m\angle BPA = 90$	(11) Reasoning similar as for statement 10
(12) $m\angle PDA + m\angle DPA = m\angle PBA + m\angle BPA$	(12) Statements 10 and 11
(13) $m\angle DPA < m\angle BPA$	(13) Statements 8 and 12. Problem 15 Chapter I
(14) $\overrightarrow{PY}$ is between $\overrightarrow{PA}$ and $\overrightarrow{PB}$ such that $\angle YPA \cong \angle DPA$	(14) Angle construction and Angle Addition postulates. Statement 13
(15) $\overrightarrow{PY}$ has a point E common with $\overline{AB}$ such that A—E—B	(15) Statement 14. Cor. to Pasch's theorem

(16) 1−1 correspondence PAE$\leftrightarrow$PAD is a congruence and $\triangle PAE \cong \triangle PAD$	(16) (*a*) $\angle YPA \cong \angle DPA$ (How ?) (*b*) $m\angle PAE = m\angle PAD = 90$ (*c*) $\overline{PA} \cong \overline{PA}$. A.S.A. Theorem
(17) $\overline{AE} \cong \overline{AD}$	(17) Corresponding parts of congruent triangles
(18) $m(\overline{AB}) > m(\overline{AE})$	(18) A−E−B. Statement 15
(19) $m(\overline{AB}) > m(\overline{AD})$	(19) Statement 17
(20) $m(\overline{AB}) > m\overline{AC}$	(20) Statements 1 and 19.

Theorem XI (7). The shortest line segment from an external point to a plane is the line segment from the point perpendicular to the plane.

Given : A plane π and point $P \notin \pi$.

To prove that : (1) Perpendicular segment from P to π is shortest line segment among all line segments from P to π.

(2) Among all line segments from P to π, the shortest is the perpendicular segment to π.

Proof.

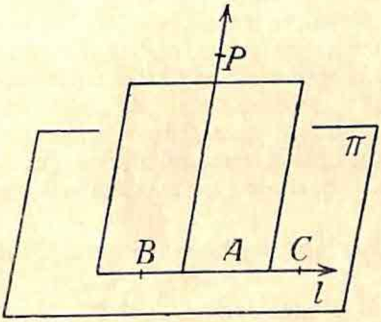


Fig. 184.

Statement	*Reason*
(1) $\overline{PA}$, the line segment perpendicular from P to π exists uniquely	(1) Theorem XI (5)
(2) An arbitrary point B belongs to the plane π	(2) Postulate 3
(3) $\overline{PA}$ and B determine a plane E	(3) Theorem : A line and a point not on the line, determine one and only one plane

(4) E and π have a line l in common through A	(4) Postulate 6 as E and π have a point A in common
(5) C is an arbitrary point on l. $C \neq A$	(5) Postulate 1
(6) In $\triangle PCA$, $m\angle PAC = 90$	(6) Statement 1 and Def. of perpendicularity of a line and a plane
(7) $m\angle PCA < 90$	(7) Sum of measures of any two angles of a triangle is less than 180. Problem 4, Chapter VIII
(8) $m\,\overline{PA} < m\,\overline{PC}$	(8) Statements 6 and 7. Theorem VIII (4)

Hence we can assert $m(\overline{PA})$ will be less than other line segments (oblique lines) from P to the plane π.

To prove the second part, assume that $\{l, l_1, l_2, l_3 \ldots\}$ are the set of all coplanar line segments from P to π such that l is the shortest line segment among all.

We want to prove that l is the perpendicular segment to π.

Let E be the plane containing the set $\{l, l_1, l_2, l_2 \ldots\}$ π and E intersect in a line q. Let r be the perpendicular line segment from P to q. If $r = l$ then we have nothing further to prove. Now assume that $r \neq l$. Then the measure of r will be less than the measure of l (First part of the theorem). But this leads to a contradiction of our assumption that l is the shortest among all the possible line segments for P to π. Hence the assumption that $r \neq l$ need to be abandoned Hence the shortest line segment from P to π is the perpendicular line segment. In fact we can consider set of planes $\{E, E_1, E_2 \ldots\}$ and all these planes wll contain the same line l which will be perpendicular from P to π.

Def. XI (3). A line and a plane are parallel iff these do not inrersect.

Def. XI (4). Two planes are parallel iff they do not intersect.

How do we know that parallel planes exist ? We have already proved that through a given point on a given line there passes one and only one plane perpendicular to the line. Also a line has atleast two distinct points (Postulate 1). So we can have atleast two distinct planes perpendicular to a given line. Will these two planes be parallel ? In this connection, let us consider the following theorem :—

Theorem XI (8). Two planes which are perpendicular to the same line are parallel.

Given : A line l. Two distinct points A and B ; $A \in l$ and $B \in l$. Plane π is $\perp$ to l at A and plane E is $\perp$ to l at B.

To prove that : $\pi \parallel E$.

Proof. There are two alternatives :

(1) Either $\pi \parallel E$ and in this case we have nothing further to prove.

(2) Or $\pi \nparallel E$. If it is so then π and E have a point C in common. Then $(C \notin l)$.

Statement	*Reason*
(1) There exists a line segment $\overline{CA}$ perpendicular to l at A	(1) π is perpendicular l. Def. of perpendicularity of a line and a plane
(2) There exists a line segment $\overline{CB}$ perpendicular to l at B	(2) $l \perp E$
(3) $\overline{CA} \neq \overline{CB}$	(3) $\pi \neq E$ and as $A \in \pi$ and $B \in E$.

From C, a point which does not lie on l, there exist two line segments perpendicular to l which leads to a contradiction of the theorem VII (6) which we have proved earlier. Hence the assumption that $\pi \nparallel E$ has to be abandoned.

Theorem XI (9). Two lines which are perpendicular to the same plane are parallel.

Given : A plane π and two lines p, q such that $p \perp \pi$ and $q \perp \pi$.

To prove that : $p \parallel q$.

Plan of Proof. In theorem IX (1), we have proved that two coplanar lines which are perpendicular to the same line are parallel. In this theorem, we are given lines p and q. Let these lines be perpendicular to π at A and B. So p, q will both be $\perp$ to the line segment $\overline{AB}$. Now if we can prove that p and q are coplanar then we would have proved the theorem at hand.

Proof.

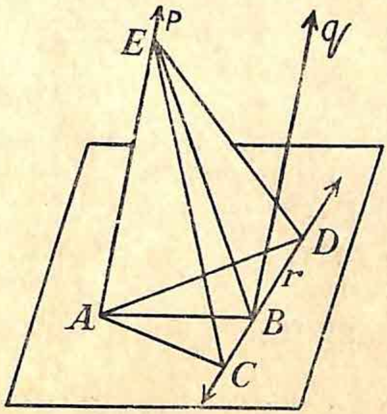


Fig. 185.

Statement	*Reason*
(1) There exists a unique line $r \perp \overline{AB}$ at B $(r \in \pi)$	(1) At a point in a line there exists a unique line which is perpendicular to given line
(2) There exists a point E on p $(E \neq A)$	(2) Postulate 1
(3) There exists points C, D each element of r such that $D-B-C$ and $\overline{BD} \cong \overline{BC}$	(3) Point plotting theorem IV (1)
(4) $m\angle ABC = m\angle ABD = 90$	(4) Statement 1
(5) $\overline{AB} \cong \overline{AB}$	(5) Any segment is congruent to itself
(6) $1-1$ correspondence $ABC \leftrightarrow ABD$ is a congruence and $\triangle ABC \cong \triangle ABD$	(6) S.A.S. Postulate
(7) $\overline{AC} \cong \overline{AD}$	(7) Corresponding parts of congruent triangles
(8) $m\angle EAD = m\angle EAC = 90$	(8) $p \perp \pi$. Def. of perpendicularity of a line and a plane
(9) $\overline{EA} \cong \overline{EA}$	(9) Why ?
(10) $1-1$ correspondence $EAC \leftrightarrow EAD$ is a congruence and $\triangle EAC \cong \triangle EAD$	(10) S.A.S. Postulate
(11) $\overline{EC} \cong \overline{ED}$	(11) Corresponding parts of congruent triangles
(12) $\overline{BD} \cong \overline{BC}$	(12) Statement 3
(13) $\overline{EB} \cong \overline{EB}$	(13) Why ?
(14) $1-1$ correspondence $EBC \leftrightarrow EBD$ is a congruence and $\triangle EBC \cong \triangle EBD$	(14) S.S.S. theorem
(15) $\angle EBD \cong \angle EBC$	(15) Corresponding parts of congruent triangles
(16) $\angle EBD$, $\angle EBC$ form a linear pair	(16) Statement 1
(17) $m\angle EBD = m\angle EBC = 90$	(17) Statement 15

(18) $r\perp EB$	(18) Statement 17
(19) $r\perp\overline{AB}$, $r\perp\overline{EB}$, $r\perp q$ and foot of perpendicular of each is B	(19) Statements 1, 18, given
(20) $\overline{AB}$, $\overline{EB}$, q are coplanar and determine a plane F	(20) Theorem XI (2)
(21) Points A, E lie in F	(21) Statement 20
(22) $\overline{AE}$ lies in F	(22) Statement 21. Postulate 5
(23) p, q lie in F and hence are coplanar.	(23) Statements 20 and 22

p, q are both $\perp$ to $\overline{AB}$ and are coplanar. Hence $p \parallel q$. [Theorem X(1)].

Theorem XI (10). If a plane intersects two parallel planes, then it intersects them in two parallel lines.

Given : Two planes P and Q such that P $\parallel$ Q. Another plane R intersects each of P and Q.

To prove that : (1) $R\cap P$ is a line ($\overleftrightarrow{AB}$ say)

(2) $R\cap Q$ is a line ($\overleftrightarrow{CD}$ say).

(3) $\overleftrightarrow{AB}\parallel\overleftrightarrow{CD}$.

Proof.

Statement	*Reason*
(1) $R\cap P$ is a line $\overleftrightarrow{AB}$	(1) Postulate 6. Since it is given that R and P intersect
(2) $R\cap Q$ is a line $\overleftrightarrow{CD}$	(2) Postulate 6. Since it is given that R and P intersect
(3) $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ donot have any point in common	(3) $\overleftrightarrow{AB}\in P$ and $\overleftrightarrow{CD}\in Q$ and since it is given that $P\parallel Q$
(4) $\overleftrightarrow{AB}\in R$ and $\overleftrightarrow{CD}\in R$	(4) Statements 1 and 2
(5) $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are coplanar	(5) Since $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ both lie in the same plane R, statement 4
(6) $\overleftrightarrow{AB}\parallel\overleftrightarrow{CD}$	(6) Statements 3 and 5

Theorem XI (11). If a plane intersects one of two parallel lines in a single point, it intersects the other in a single point also.

Given : a plane π and two lines l_1 and l_2 such that $l_1 \parallel l_2$. π intersects l_1 in a point A.

To prove that : π intersects l_2 in a single point B (say).

Plan of Proof. To prove that l_2 and π intersect in a single point, we have to prove that :

(1) π and l_2 intersect or $l_2 \nparallel \pi$.

(2) l_2 does not wholly lie in π or $l_2 \nsubseteq \pi$.

Proof.

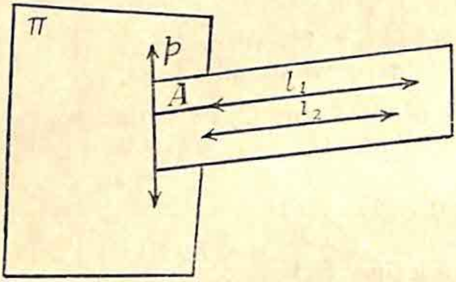


Fig. 186.

Statement	*Reason*
(1) Plane E contains both l_1 and l_2	(1) Since $l_1 \parallel l_2$ and hence these are coplanar
(2) l_1 and π have a common point A	(2) Given
(3) E and π intersect	(3) E and π have a common point A. (Statement 2)
(4) Intersection of E and π is a line p through A	(4) Postulate 6
(5) $p \neq l_1$ (p is not same as l_1)	(5) If $p = l_1$ then intersection of π and l_1 will be more than one point
(6) $p \neq l_2$ (p is not same as l_2)	(6) $l_1 \parallel l_2$ (given) and as p and l_1 intersect
(7) $l_2 \nparallel p$	(7) If $p \parallel l_2$ and as we know that $l_1 \parallel l_2$ and as p and l_1 intersect, it would imply that two intersecting lines p and l_1 are both parallel to l_2 which contradicts parallel Postulate

(8) l_2 and p intersect in a single point	(8) Statements 6 and 7 and as l_2 and p are coplanar (Statements 3 and 4)
(9) $l_2 \nparallel \pi$	(9) Statement 8
(10) l_2 does not wholly lie in π	(10) as $l_1 \parallel l_2$ and l_1, l_2 are coplanar and l_1 has only one point common with π. Also statement 6
(11) l_2 and π intersect in a single point.	(11) Statements 8, 9 and 10.

Theorem XI (12). If a line intersects one of two parallel planes in a single point, it intersects the other in a single point also.

Given : Two planes π_1 and π_2 such that $\pi_1 \parallel \pi_2$. A line l intersects π_1 in a point A.

To prove that : l intersects π_2 in a single point.

Plan of Proof. To prove that l intersects π_2 in a single point, we must prove that l does not wholly lie in π_2 ($l \nsubseteq \pi_2$) Postulate 5 and also $l \nparallel \pi_2$.

Proof.

Statement	*Reason*
(1) l intersects π_1 in A	(1) Given
(2) $\pi_1 \parallel \pi_2$	(2) Given
(3) l does not wholly lie in π_2	(3) If l wholly lies in π_2 and as l has a point A in common with π_1 then π_1 cannot be parallel to π_2.

Now we want to prove that $l \nparallel \pi_2$. Let us assume that $l \parallel \pi_2$ and then prove that it leads to contradiction of some postulate or some theorem that we have proved. So we proceed with the assumption that $l \parallel \pi_2$

Statement	*Reason*
(4) There exists a point B in π_2	(4) Postulate 3
(5) l and B determine a plane E	(5) Theorem : A line and a point not on the line exactly determine one plane.
(6) E and π_2 intersect	(6) Statements 4 and 5
(7) E and π_2 intersect in a line through B (l_2 say)	(7) Postulate 6

(8) E and π_1 intersect	(8) l lies in E and l and π_1 have a point A in common
(9) E and π_1 intersect in a line say l_1 through the point A.	(9) Postulate 6

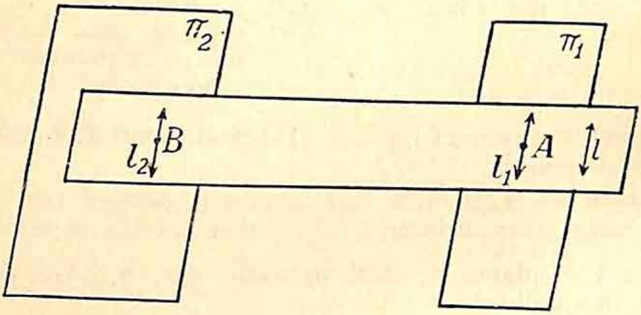


Fig. 187.

(10) $l_1 \parallel l_2$	(10) Theorem XI (10)
(11) $l \parallel l_2$	(11) Assumption : $l \parallel \pi_2$ and as l and l_2 are coplanar (lying in E)
(12) l_1 contains A	(12) Statement 9
(13) l contains A	(13) Given
(14) l and l_1 intersect in A	(14) Statements 12 and 13
(15) Two intersecting lines l and l_1 are both parallel to the same line l_2.	(15) Statements 10, 11, 14.

The statement 15, contradicts the parallel postulate as l, l_1 and l_2 are coplanar. Hence the assumption that $l \parallel \pi_2$ need to be abandoned. Therefore l neither wholly lies in π_2 nor it is parallel to π_2. Hence l intersects π_2 in one point only.

Theorem XI (13). If a line is perpendicular to one of two parallel planes, it is perpendicular to the other.

Given : Two planes π_1 and π_2 such that $\pi_1 \parallel \pi_2$. A line l is perpendicular to π_1 at A.

To prove that : $l \perp \pi_2$ (at a point say B).

Plan of Proof. To prove that $l \perp \pi_2$, we must prove that l intersects π_2 in a single point and l is perpendicular to every line in π_2 which passes through that point.

Proof. Let l_2 be a line lying in π_2. Let C be any point of l_2. Let $\overline{CD}$ be perpendicular from C to π_1 having a point D in common with π_1. If D=A, then $\overleftrightarrow{CD}=l$ as through a given point in a given plane, there is atmost one line $\perp$ to the given plane. [Theorem XI (4)]. Also then l will intersect π_2 in a single point. [Theorem XI (12)].

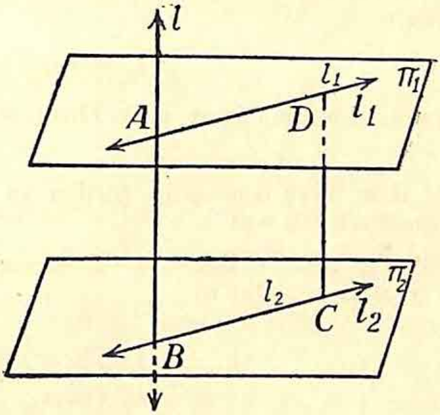


Fig. 188.

If D$\neq$A then $\overline{CD} \parallel l$ [Theorem XI (9)]. Now $l \nparallel l_2$ as otherwise then l_2 will be parallel to $\overline{CD}$ and as l_2 and $\overline{CD}$ intersect at C it will contradict the parallel postulate. Hence l and l_2 intersect at a single point (B say). Hence in both the cases l intersects π_2 in a single point.

Now we want to prove that every line in π_2 which contains B is $\perp l$. Let π be the plane that contains l_2 and l. Then π intersects π_1 in l_1. Hence $l_1 \parallel l_2$ [Theorem XI (10)]. But $l_1 \perp l$ [Theorem XI (1). Hence $l_2 \perp l$. (Why ?)]

Theorem XI (14). If a plane is perpendicular to one of two parallel lines, it is perpendicular to the other line also.

Given : Two lines p and q such that $p \parallel q$. Also a plane π such that $\pi \perp p$ at A.

To prove that : $\pi \perp q$ at a single point B.

Plan of Proof. We will assume that q is not $\perp \pi$ at the only point at which π intersects q [Theorem XI (11). Then we will try to prove the existence of a line perpendicular to π at that point which will be parallel to p. This situation will lead to contradiction of parallel postulate as this will imply the existence of two intersecting lines being parallel to the same line.

Proof.

Statement	*Reason*
(1) p and π intersect at a point A	(1) Given
(2) $p \parallel q$	(2) Given
(3) q and π must intersect at a single point B.	(3) Theorem XI (11)

If $q \perp \pi$ at B, then there is nothing further to prove. Assume that q is not perpendicular to π at B.

(4) There exists a unique line r which is perpendicular to π at B	(4) Theorem XI (4)
(5) $r \parallel p$	(5) Theorem XI (9)
(6) $q \parallel p$	(6) Given.

The statements 5 and 6 imply that two lines r and q which have a common point B are parallel to the same line p and this contradicts the parallel postulate. Hence the assumption that q is not perpendicular to π at B has to be abandoned. Of course the lines p, q, r are coplanar.

Theorem XI (15). There is a unique plane which is parallel to a given plane and also contains a given point not in the given plane.

Given : A plane π and a point A which does not lie in π.

To prove that : There is a unique plane E through A such that $E \parallel \pi$.

Plan of Proof. With the help of theorems XI (1), XI (2), XI (5), XI (8) etc. we will prove that there exists atleast one plane through A and parallel to π. Then to prove that there exists only one such plane, we will assume the existence of two such planes. Then we will prove that this leads to a contradiction of parallel postulate.

Proof.

Statement	*Reason*
(1) There exists a unique line l through A such that $l \perp \pi$	(1) Theorem XI (5)
(2) There exists a unique plane E through A such that $E \perp l$	(2) Theorem XI (3)
(3) $E \parallel \pi$	(3) Theorem XI (8)

Now to prove that E is the only plane parallel to π, we assume the contrary and state that there exists another plane F through A such that F $\parallel \pi$.

(4) E and F intersect in a line m	(4) Postulate 6
(5) There exists a point B in π and through B there exists a line n not parallel to m	(5) Parallel postulate. Since there exists atmost one line through B parallel to m
(6) Point A and the line n determine a plane G	(6) Theorem : A line and a point not in the line determine a unique plane
(7) G does not contain the line m	(7) Statements 5 and 6
(8) G has a point A in common with each of the planes E and F	(8) Statement 6
(9) G intersects the plane E in a line p and the plane F in the line q	(9) Postulate 6
(10) p, q have only point A in common	(10) Statements 7, 9

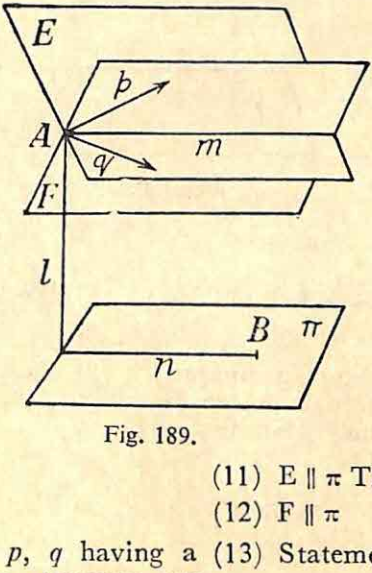


Fig. 189.

(11) $p \parallel n$	(11) E $\parallel \pi$ Theorem XI (10)
(12) $q \parallel n$	(12) F $\parallel \pi$
(13) Two lines p, q having a point A in common are each parallel to m.	(13) Statements 10, 11, 12

The statement 13 leads to a contradiction of the parallel postulate. Hence the assumption that F $\parallel \pi$ needs to be abandoned.

Theorem XI (16). Two planes which are parallel to a third plane are parallel to each other.

Given : Planes π_1, π_2 and π_3 such that :

$$\pi_1 \parallel \pi_2 \text{ and } \pi_1 \parallel \pi_3$$

To prove that : $\pi_2 \parallel \pi_3$.

Plan of Proof. A line $p \perp \pi$, at A will be $\perp \pi_2$ and also will be $\perp \pi_3$ [Theorem XI (13)]. Then we will apply theorem XI (8) to prove that $\pi_2 \parallel \pi_3$.

Proof

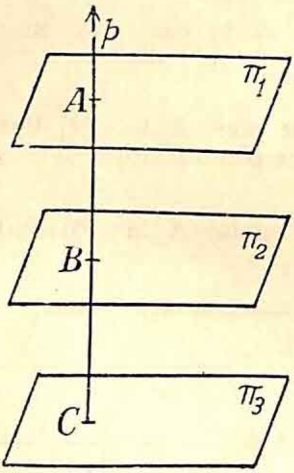


Fig. 190.

Statement	*Reason*
(1) There exists a point A in π_2	(1) Postulate 3
(2) There exists a unique line p through A which is perpendicular to π_1 ($p \notin \pi_1$)	(2) Theorem XI (4)
(3) p intersects π_2 in a single point B	(3) Theorem XI (12)
(4) p intersects π_3 in a single point C	(4) Theorem XI (12)

(5) $p \perp \pi_2$	(5) Theorem XI (13)
(6) $p \perp \pi_3$	(6) Theorem XI (13)
(7) $\pi_2 \parallel \pi_3$	(7) Statements 5 and 6. Theorem XI (8).

Perpendicular Planes. We have talked about a line perpendicular to a plane or two planes being parallel. When are two planes perpendicular to each other? For this we have to make another definition—that of a dihedral angle.

Definition XI (5). (*Dihedral angle*). It is the union of a line and two half planes. Two half planes have the line as their edge and do not lie in the same plane.

In this figure $\overleftrightarrow{AB}$ is the line. There is a half plane determined by P and $\overleftrightarrow{AB}$ and another half plane determined by Q and $\overleftrightarrow{AB}$. Of course Q does not lie in the plane determined by P and $\overleftrightarrow{AB}$.

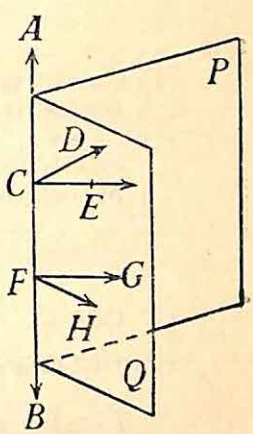


Fig. 191.

This dihedral angle will be denoted by $\angle P-AB-Q$. The common line of the two half planes is called the edge of the dihedral angle. So $\overleftrightarrow{AB}$ is the edge of dihedral $\angle P-AB-Q$. The two half planes are called the faces of the dihedral angle.

Another concept with which we should be familiar is that of a plane angle of a dihedral angle.

Definition XI (6). A plane angle of a dihedral angle is the angle determined by two rays in which any plane perpendicular to the edge of the dihedral angle intersects the faces of the dihedral angle.

A dihedral angle has many plane angles. In this figure $\angle DCE$, $\angle GFH$ etc. etc. are the plane angles of the dihedral angle. We can prove that all plane angles of a dihedral angle are congruent. The measure of a dihedral angle is taken to be same as the measure of its plane angle. This statement assumes that all plane angles of a dihedral angle should be congruent. We can, however, prove it as a theorem.

Theorem XI (17). Any two plane angles of any given dihedral angle are congruent.

Given : A dihedral angle $\angle P-AB-Q$. Any two plane angles $\angle CDE$ and $\angle FGH$ of the same dihedral angle (Fig. 192).

To prove that : $\angle CDE \cong \angle FGH$.

Proof.

Statement	*Reason*
(1) On $\overrightarrow{GF}$, there exists a point L such that $\overline{GL}\cong\overline{DC}$ and on $\overrightarrow{GH}$ there exists a point M such that $\overline{GM}\cong\overline{DE}$	(1) Segment construction theorem
(2) $\overline{CD} \parallel \overline{GL}$	(2) $\overline{CD}\perp\overline{AB}$ and $\overline{GL}\perp\overline{AB}$ Theorem IX (1)
(3) $\overline{CD}\cong\overline{GL}$	(3) Statement 1
(4) DCLG is a $\parallel^{gm}$	(4) Theorem IX (18)
(5) $\overline{CL}\cong\overline{DG}$ and $\overline{CL} \parallel \overline{DG}$	(5) Theorem IX (13)
(6) $\overline{DE} \parallel \overline{GM}$	(6) Reason as for statement 2
(7) $\overline{DE}\cong\overline{GM}$	(7) Statement 1
(8) DEMG is a $\parallel^{gm}$	(8) Why ?

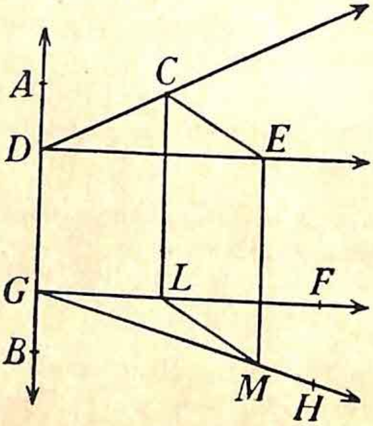


Fig. 192.

(9) $\overline{EM} \cong \overline{DG}$ and $\overline{EM} \parallel \overline{DG}$	(9) Why ?
(10) $\overline{CL} \cong \overline{EM}$	(10) Statements 5 and 9
(11) $\overline{CL} \parallel \overline{EM}$	(11) Chapter IX. Problem 1. Statements 5 and 9.
(12) CLME is a $\parallel^{gm}$	(12) Theorem IX (18)
(13) $\overline{CE} \cong \overline{LM}$	(13) Theorem IX (13)
(14) 1—1 correspondence $CDE \leftrightarrow LGM$ is a congruence and $\triangle CDE \cong \triangle LGM$	(14) S.S.S. Theorem
(15) $\angle CDE \cong \angle LGM$	(15) Corresponding parts of congruent angles
(16) $\angle CDE \cong \angle FGH$	(16) Why ?

We have already noted that the measure of a plane angle of a dihedral angle is taken to be the measure of the dihedral angle. A dihedral angle whose measure is 90 is called a right dihedral angle. We are now in a position to define perpendicular planes.

Definition XI (7). (*Perpendicular planes*). Two planes are said to be perpendicular if they determine a right dihedral angle.

Theorem XI (18). If a line is perpendicular to a plane, then any plane containing the given line is perpendicular to the given plane.

Given : A line p perpendicular to plane π at a point A.

To prove that : A plane G containing the line p is perpendicular to the plane π.

Proof.

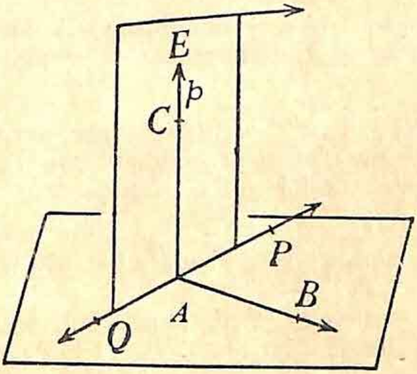


Fig. 193.

Statement	*Reason*
(1) There exists a line $\overleftrightarrow{AB}$ through A. ($\overleftrightarrow{AB} \in \pi$)	(1) Postulates 1 and 3
(2) There exists a unique line $\overleftrightarrow{PQ}$ through A such that $\overleftrightarrow{PQ} \perp \overleftrightarrow{AB}$	(2) Theorem VI (3)
(3) p and $\overleftrightarrow{PQ}$ determine a unique plane E	(3) Theorem : Two intersecting lines determine exactly one plane.
(4) There exists a point C on p	(4) Why ?
(5) $m\angle CAB = 90$	(5) $p \perp \pi$ (given). Hence $p \perp \overleftrightarrow{AB}$ (Definition of perpendicularity of a line and a plane)
(6) $\angle CAB$ is a plane angle of dihedral angle $\angle C-PQ-B$	(6) Definition of a dihedral angle
(7) $\angle C-PQ-B$ is a right dihedral angle	(7) Statements 5 and 6
(8) E and π are perpendicular to each other	(8) Statement 7

Since $\overleftrightarrow{AB}$ is any arbitrary line through A and E is any arbitrary plane containing p, the statement 8 establishes the proof of the theorem.

Theorem XI (19). If two planes are perpendicular then a line lying in one of the planes, perpendicular to their line of intersection will be perpendicular to the other plane.

Given : Two planes E and π such that $E \perp \pi$. A line $\overleftrightarrow{AC} \in E$ such that $\overleftrightarrow{AC} \perp \overleftrightarrow{PQ}$ where $\overleftrightarrow{PQ}$ is the line of intersection of E and π. [Refer to figure of theorem XI (18)].

To prove that : $\overleftrightarrow{AC} \perp \pi$.

Proof.

Statement	*Reason*
(1) There exists $\overrightarrow{AB} \perp \overline{PQ}$ where $\overrightarrow{AB} \in \pi$	(1) Theorem VI (3)
(2) $\angle CAB$ is a plane angle of the dihedral angle $\angle C{-}PQ{-}B$	(2) Definition of a dihedral angle
(3) $m\angle CAB = 90$	(3) Given that $E \perp \pi$
(4) $\overrightarrow{AC} \perp \overrightarrow{AB}$	(4) Statement 3
(5) $\overrightarrow{AC} \perp \overleftrightarrow{PQ}$ as well as $\overrightarrow{AC} \perp \overrightarrow{AB}$	(5) Given. Statement 4
(6) $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ both belong to π	(6) Given. Statement 1
(7) $\overrightarrow{AC} \perp \pi$	(7) Statements 5 and 6, Theorem XI (1)

Theorem XI (**20**). If two intersecting planes are each perpendicular to a third plane, their line of intersection is perpendicular to that plane.

Given : Two planes P and Q having $\overleftrightarrow{AB}$ as their line of intersection and another plane π such that $P \perp \pi$ and $Q \perp \pi$.

To prove that : $\overleftrightarrow{AB} \perp \pi$.

Proof.

Statement	*Reason*
(1) $\overleftrightarrow{CD}$ is the line of intersection of P and π and $\overleftrightarrow{CG}$ is the line of intersection of Q and π	(1) Given that $P \perp \pi$ and $Q \perp \pi$
(2) $\overrightarrow{CK}$ is a line perpendicular to $\overrightarrow{CD}$ such that $\overrightarrow{CK} \in P$	(2) Theorem VI (3)

(3) $\overrightarrow{CK'}$ is the line perpendicular to $\overrightarrow{CG}$ such that $\overrightarrow{CK'} \in Q$	(3) Theorem VI (3)
(4) $\overrightarrow{CK}$ and $\overrightarrow{CK'}$ are both perpendicular to π.	(4) Theorem XI (19)

If $\overrightarrow{CK} = \overrightarrow{CK'}$ and each of $\overrightarrow{CK}$, $\overrightarrow{CK'}$ belongs to $\overrightarrow{AB}$, then there is nothing further to prove. Assume $\overrightarrow{CK} \neq \overrightarrow{CK'}$ but this leads us to contradiction of theorem XI (4). Hence the assumption that $\overrightarrow{CK} \neq \overrightarrow{CK'}$ has to be abandoned. Now $\overrightarrow{CK}$ or $\overrightarrow{CK'}$ belong to both P and Q and as $\overleftrightarrow{AB}$ is the unique line of intersection of P and Q. Hence $\overrightarrow{CK}$ or $\overrightarrow{CK'}$ belong to $\overleftrightarrow{AB}$. Hence, $\overleftrightarrow{AB}$ is perpendicular to π.

Problem Set

I. At a given point A on a line p, how many lines are perpendicular to l in a two dimensional space ? In a three dimensional space ?

II. A line p is not perpendicular to a plane π but it is perpendicular to a line l that lies in the plane π at a point A. How many more lines will be perpendicular to l at A ?

III. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. $\overleftrightarrow{CD}$ lies in a plane π. Will $\overleftrightarrow{AB}$ be always parallel to π ?

IV. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Prove that a plane π containing any one of $\overleftrightarrow{AB}$ or $\overleftrightarrow{CD}$ will be parallel to the other line.

V. If a line is parallel to a plane, it is parallel to the intersection of that plane with any plane containing that line.

VI. Can two intersecting lines be parallel to a plane ? If so under what condition ? Prove that the plane containing the two intersecting lines will be parallel to the given plane.

CHAPTER XII

PROPORTION AND SIMILARITY

The geometric concept of similarity is very closely related to the concept of proportionality in Arithmetic. The geometric concept of similarity was developed by Greeks at a time when the concept of ratio and proportion was unknown to them. Consequently the theorems based on concept of similarity were considered by them to be a very difficult portion of geometry. This topic, however, is easy to learn if we make use of the concept of proportionality that we generally study in Arithmetic. Therefore before we attempt to define the geometric concept of similarity, it will be appropriate to briefly discuss the concept of Ratio and Proportion.

Definiton XII (1). If a, b are any two real numbers (or if $a, b \in \mathrm{R}$) then $\frac{a}{b}$ is called the ratio of a to b.

The ratio of a to b is denoted by $a : b$ or $\frac{a}{b}$

Definition XII (2) (*Proportion*). A true statement of equality between two ratios is called a proportion.

$\frac{a}{b}=\frac{c}{d}$ is a proportion ; $\frac{3}{5}=\frac{6}{10}$ is a proportion.

$\frac{3}{5}=\frac{7}{10}$ is not a proportion. $\frac{a}{b}=\frac{c}{d}$ is also written as $a : b : : c : d$. The four quantities that form a proportion are called its terms. In $a : b : : c : d$, a is called the first term, b the second term, c the third term and d the fourth term. The second and third terms are known as means while the first and fourth terms are called extremes. If we are given the numbers a, b, c then d is the fourth proportional if $\frac{a}{b}=\frac{c}{d}$. If the means of a proportion are equal then either of these will be the mean porportional between the first and fourth. For example if $a : b=b : c$, then b will be the mean proportional between a and c. Also in this situation c will be the third proportional to a and b.

In general if we are given two sequences $\{a, b, c...\}$ and $\{a', b', c'...\}$ and if it is true that $\frac{a'}{a}=\frac{b'}{b}=\frac{c'}{c}$ ... then we say that the two squences are proportional and we write that

$$a, b, c \ldots \sim a', b', c' \ldots$$

Theorem XII (1). In a proportion, the product of the extremes equals the product of the means.

Given : $\frac{a}{b}=\frac{c}{d}$

To Prove that : $ad=bc$

Proof.

Statement	*Reason*
(1) $\frac{a}{b}=\frac{c}{d}$	(1) given.
(2) $bd\times\frac{a}{b}=bd\times\frac{c}{d}$	(2) why ?
(3) $da=bc$	(3) why ?
(4) $ad=bc$	(4) why ?

Theorem XII (2). In a series of equal ratios, the ratio of the sum of numerators to the sum of denominators equals any one of the given ratio.

Given : $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}\ldots$

To Prove that : $\frac{a+c+e+\ldots}{b+d+f+\ldots}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}\ldots$

Proof.

Statement	*Reason*
(1) If $\frac{a}{b}=r$ then $\frac{c}{d}=r,\ \frac{e}{f}=r\ldots$	(1) Why ?
(2) $a=br,\ c=dr,\ e=fr\ldots$	(2) Why ?
(3) $a+c+e\ldots=br+dr+fr\ldots$	(3) Why ?
(4) $a+c+e\ldots=(b+d+f\ldots)\ r$	(4) Why ?
(5) $\frac{a+c+e\ldots}{b+d+f\ldots}=r$	(5) Why ?
(6) $\frac{a+c+e\ldots}{b+d+f\ldots}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}\ldots$	(6) Why ?

Theorem XII(3). In a proportion, the means may be interchanged. *(Alternendo)*

Given : $\frac{a}{b}=\frac{c}{d}$

To Prove that : $\frac{a}{c}=\frac{b}{d}$

Proof.

Statement	*Reason*
(1) $\frac{a}{b}=\frac{c}{d}$	(1) given
(2) $ad=bc$	(2) Why ?
(3) $\frac{ad}{cd}=\frac{bc}{cd}$	(3) Why ?
(4) $\frac{a}{c}=\frac{b}{d}$	(4) Why ?

Theorem XII (**4**). In a proportion, the ratios may be inverted. (*Invertendo*)

Given : $\frac{a}{b}=\frac{c}{d}$

To Prove that : $\frac{b}{a}=\frac{d}{c}$

Proof.

Statement	*Reason*
(1) $\frac{a}{b}=\frac{c}{d}$	(1) given
(2) $ad=bc$	(2) Why ?
(3) $bc=ad$	(3) Why ?
(4) $\frac{bc}{ac}=\frac{ad}{ac}$	(4) Why ?
(5) $\frac{b}{a}=\frac{d}{c}$	(5) Why ?

Theorem XII (**5**). If $\frac{a}{b}=\frac{c}{d}$ then $\frac{a+b}{b}=\frac{c+d}{d}$ and $\frac{a-b}{b}=\frac{c-d}{d}$ (*Componendo, Dividendo*)

Proof.

Statement	*Reason*
(1) $\frac{a}{b}=\frac{c}{d}$	(1) Why ?

(2) $\frac{a}{b}+1=\frac{c}{d}+1$	(2) Why ?
(3) $\frac{a+b}{b}=\frac{c+d}{d}$	(3) Why ?
(4) $\frac{a}{b}-1=\frac{c}{d}-1$	(4) Why ?
(5) $\frac{a-b}{b}=\frac{c-d}{d}$	(5) Why ?

Cor. If $\frac{a}{b}=\frac{c}{d}$ then $\frac{a}{a+b}=\frac{c}{c+d}$

$$\left(\text{If } \frac{a}{b}=\frac{c}{d} \text{ then } \frac{b}{a}=\frac{d}{c} \text{ or } \frac{b+a}{a}=\frac{d+c}{c} \leftrightarrow \frac{a}{a+b}=\frac{c}{c+d}\right)$$

Theorem XII (6). If $\frac{a}{b}=\frac{c}{d}$ then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$ (*Componendo and Dividendo*).

Proof.

Statement	*Reason*
(1) $\frac{a}{b}=\frac{c}{d}$	(1) given
(2) $\frac{a+b}{b}=\frac{c+d}{d}$	(2) Theorem XII (5)
(3) $\frac{a-b}{b}=\frac{c-d}{d}$	(3) Theorem XII (5)
(4) $\frac{a+b}{b}\div\frac{a-b}{b}=\frac{c+d}{d}\div\frac{c-d}{d}$	(4) Why ?
(5) $\frac{a+b}{a-b}=\frac{c+d}{c-d}$	(5) Statement 4.

Continued Proportion : If $a : b=b : c$, then a, b, c are said to be in continued proportion. In this case b will be the mean proportion between a and c and c is called the third proportion. a, b, c, d will be in continued proportion if $a : b=b : c=c : d$ etc. etc.

Definition XII (3). Two sequences of numbers $a, b, c...$ and $p, q, r...$such that none of $a, b, c...$ or $p, q, r...$ equals zero are said to be proportional if

$$\frac{a}{p}=\frac{b}{q}=\frac{c}{r} \dots \text{ or } \frac{p}{a}=\frac{q}{b}=\frac{r}{c} \dots$$

Now we prove a very important theorem involving the concept of proportionality. The concept of similarity that we are going to develop later is very much based on this theorem.

Theorem XII (7) If a line parallel to one side of a triangle intersects its other two sides in distinct points which are interior points of those sides then the lengths of the segments into which one side is divided are proportional to the lengths of corresponding segments of the other side.

Given : A $\triangle ABC$ and a point D in the interior of the side $\overline{AB}$. A line $\overleftrightarrow{DE} \parallel \overline{BC}$ has a point E in common with $\overline{AC}$ such that A−E−C.

To Prove that : $\frac{m\,(\overline{BD})}{m\,(\overline{AD})}=\frac{m\,(\overline{EC})}{m\,(\overline{AE})}$ or $\frac{BD}{AD}=\frac{EC}{AE}$

[It may be noted that we have chosen to denote $m\,(\overline{BD})$ by BD etc. etc.]

Proof.

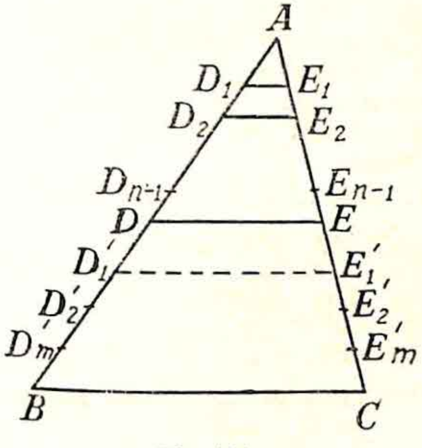


Fig. 194.

Statement	*Reason*
(1) $\overline{DE} \parallel \overline{AC}$ meets $\overline{AC}$ in its interior point	(1) Given, Pasch's theorem
(2) On $\overline{AD}$, there are $n-1$ points $D_1, D_2, \ldots D_{n-1}$ such that $A-D_1-D_2\ldots D_{n-1}-D$ and $\overline{AD_1}\cong\overline{D_1D_2}\cong\ldots\cong\overline{D_{n-1}D}$ $=\frac{AD}{n}=\Sigma$. (In non technical words we have divided $\overline{AD}$ into n equal parts by $n-1$ points)	(2) point plotting theorem (It may be noted that Σ will be a very very small number)

(3) On $\overline{DB}$, there are m points $D'_1, D'_2 \ldots D'_m$ such that $D-D'_1-D'_2 \ldots D'_m-B$ and $\overline{DD'_1} \cong \overline{D'_1D'_2} \cong \ldots \cong D'_{m-1}D'_m = \Sigma$ and $D'_mB = r \leqslant \Sigma$	(3) Point Plotting theorem.

(In non-technical words, we have divided $\overline{DB}$ into a number of parts whose length equals $\frac{AD}{n}$ or Σ and we get m parts while a small segment $\overline{D'_mB}$ is left out. The length of D'_mB is less than Σ. At the most its length may be equal to Σ).

(4) Lines through $D_1, D_2 \ldots D_{n-1}$ each parallel to $\overline{BC}$ intersect $\overline{AE}$ in points $E_1, E_2, \ldots, E_{n-1}$ which are its interior points. Also lines through $D'_1, D'_2, \ldots, D'_m$ each parallel to $\overline{BC}$ intersect $\overline{EC}$ in points $E'_1, E'_2 \ldots, E'_m$ which are interior points of $\overline{EC}$	(4) Parallel Postulate. Pasch's theorem. (It may be noted that ordering of points on $\overline{AC}$ would be $A-E_1-E_2 \ldots E_{n-1}-E-E'_1-E'_2 \ldots E'_m-C$. Any disturbance in this order will contradict the parallel postulate)
(5) $AE_1 = E_1E_2 = \ldots E_{n-1}E = \frac{AE}{n} = \Sigma'$ (Σ' will be a very very small number and it may be different from Σ)	(5) Theorem IX (20)
(6) $EE'_1 = E'_1E'_2 = \ldots E'_{m-1}E'_m = \Sigma'$ and $E'_mC = r'$ where $r' \leqslant \Sigma'$ (r' may be different from r)	(6) Theorem IX (20)
(7) $AD = n\Sigma$, $DB = m\Sigma + r$ $AE = n\Sigma'$, $EC = m\Sigma' + r'$	(7) Statements 2, 3, 5, 6
(8) $\frac{DB}{AD} = \frac{m\Sigma + r}{n\Sigma} = \frac{m}{n} + \frac{r}{n\Sigma} = \frac{m}{n} + \frac{r}{AD}$ and $\frac{EC}{AE} + \frac{m\Sigma' + r'}{n\Sigma'} = \frac{m}{n} + \frac{r'}{n\Sigma'} = \frac{m}{n} + \frac{r'}{AE}$	(8) Statement 7

(9) $\left\|\frac{\mathrm{DB}}{\mathrm{AD}}-\frac{\mathrm{EC}}{\mathrm{AE}}\right\|=\left\|\left(\frac{m}{n}+\frac{r}{\mathrm{AD}}\right)-\left(\frac{m}{n}+\frac{r'}{\mathrm{AE}}\right)\right\|=\left\|\frac{r}{\mathrm{AD}}-\frac{r'}{\mathrm{AE}}\right\|$	(9) Statement 8
(10) $r\leqslant\Sigma$ and $r'\leqslant\Sigma'$	(10) Statements 3 and 6.
(11) $\Sigma=\frac{\mathrm{AD}}{n}$ and $\Sigma'=\frac{\mathrm{AE}}{n}$	(11) Statements 2 and 3
(12) $\frac{r}{\mathrm{AD}}\leqslant\frac{1}{n}$ and $\frac{r'}{\mathrm{AE}}\leqslant\frac{1}{n}$	(12) Statements 10 and 11
(13) $\left\|\frac{r}{\mathrm{AD}}-\frac{r'}{\mathrm{AE}}\right\|<\frac{r}{\mathrm{AD}}+\frac{r'}{\mathrm{AE}}\leqslant\frac{2}{n}$	(13) Statement 12. Problem 15 Chapter I
(14) $\left\|\frac{r}{\mathrm{AD}}-\frac{r'}{\mathrm{AE}}\right\|=0$	(14) Statement 13. Since Σ is very very small, n is very very large. We can chose n to be as large as we please. Hence $\frac{2}{n}$ nearly equals zero and the only way in which $\left\|\frac{r}{\mathrm{AD}}-\frac{r'}{\mathrm{AE}}\right\|$ can be less than $\frac{2}{n}$ is that it equals zero. Also $\left\|\frac{r}{\mathrm{AD}}-\frac{r'}{\mathrm{AE}}\right\|$ can never be less than zero (why ?)
(15) $\left\|\frac{\mathrm{DB}}{\mathrm{AD}}-\frac{\mathrm{EC}}{\mathrm{AE}}\right\|=0$	(15) Statements 9 and 14
(16) $\frac{\mathrm{DB}}{\mathrm{AD}}=\frac{\mathrm{EC}}{\mathrm{AE}}$	(16) Statement 15.

Theorem XII (8). If a line parallel to one side of a triangle intersects the other two sides in distinct points in their interior, then it cuts off segments which are proportional to these sides.

Given : A$\triangle$ABC. A point D in the interior of $\overline{\mathrm{AB}}$. A line $\overleftrightarrow{\mathrm{DE}} \parallel \overline{\mathrm{BC}}$ has a point E in common with $\overline{\mathrm{AC}}$ such that A$-$E$-$C (Fig. 194).

To prove that : (1) $\frac{\mathrm{AD}}{\mathrm{AB}}=\frac{\mathrm{AE}}{\mathrm{AC}}$ and $\frac{\mathrm{DB}}{\mathrm{AB}}=\frac{\mathrm{EC}}{\mathrm{AC}}$.

Proof : We know that $\frac{DB}{AD}=\frac{EC}{AE}$ [(Theorem XII (7)]

Hence $\frac{DB+AD}{AD}=\frac{EC+AE}{AE}$ (*Componendo*)

or $\frac{AB}{AD}=\frac{AC}{AE} \leftrightarrow \frac{AD}{AB}=\frac{AE}{AC}$ (*invertendo*)

Also $\frac{DB}{AD}=\frac{EC}{AE} \leftrightarrow \frac{AD}{DB}=\frac{AE}{EC}$ (*invertendo*)

$\leftrightarrow \frac{AD+DB}{DB}=\frac{AE+EC}{EC} \leftrightarrow \frac{AB}{DB}=\frac{AC}{EC}$

$\leftrightarrow \frac{DB}{AB}=\frac{EC}{AC}$.

Theorem XII (9). [*Converse to theorem XII (7)*] : If a line intersects any two sides of a triangle in distinct points which lie in their interior such that the lengths of the segments into which one side is divided are proportional to the lengths of corresponding segments of the other side then the given line is parallel to the third side.

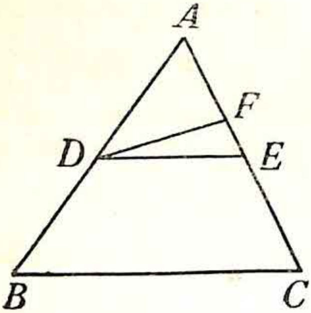


Fig. 195.

Given : A $\triangle ABC$ and a line l intersecting $\overline{AB}$, $\overline{AC}$ in points D, E such that A—D—B and A—E—C and $\frac{DB}{AD}=\frac{EC}{AE}$.

To Prove that : $l \parallel \overline{BC}$.

Proof. There are two alternatives :

(1) $l \parallel \overline{BC}$ and then we have nothing further to prove.

(2) $l \nparallel \overline{BC}$ then :

Statement	*Reason*
(1) Through D, there exists a line $\overleftrightarrow{DF} \parallel \overline{BC}$ meeting $\overline{AC}$ in F such that A—F—C and F$\neq$E	(1) Parallel Postulate, Pasch's theorem.
(2) $\frac{DB}{AD}=\frac{FC}{AF}$	(2) Theorem XII (7)

(3) $\frac{DB}{AD}=\frac{EC}{AE}$	(3) Given
(4) $\frac{FC}{AF}=\frac{EC}{AE}$	(4) Statements 2 and 3
(5) $\frac{FC+AF}{AF}=\frac{EC+AE}{AE}$	(5) Statement 5, componendo
(6) $\frac{AC}{AF}=\frac{AC}{AE}$	(6) Statement 5. Def. of betweenness, Statement 1.
(7) $AF=AE$	(7) Statement 6.
(8) E and F are both on the C-side of A.	(8) Given. Statement 1.

Since E and F are distinct points, statements 8 and 9 lead to a contradiction of the Ruler Postulate. Hence the assumption that $l \nparallel \overline{BC}$ needs to be abandoned.

Theorem XII (10). A bisector of one angle of a triangle divides the opposite side into segments which are in the ratio of the sides containing the angle.

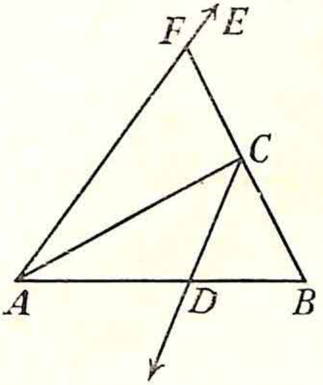


Fig. 196.

Given : A $\triangle ABC$. $\overrightarrow{CD}$ is the angle bisector of $\angle C$, having D in common with $\overline{AB}$,

To prove that : $\frac{AD}{DB}=\frac{AC}{CB}$.

Proof.

Statement	*Reason*
(1) Through A, there is a line $\overleftrightarrow{AE} \parallel \overleftrightarrow{CD}$	(1) Parallel Postulate

(2) $\overleftrightarrow{AE}$ has a point F in common with $\overleftrightarrow{BC}$	(2) $\overleftrightarrow{AE} \nparallel \overleftrightarrow{BC}$ as otherwise it will contradict parallel postulate because $\overleftrightarrow{AE} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{CD}$ and $\overleftrightarrow{CB}$ have a point C in common. Hence $\overleftrightarrow{AE}$ and $\overleftrightarrow{BC}$ intersect.
(3) F is not between C and B	(3) If C—F—B then $\overrightarrow{AF}$ will be between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (Cor. to Pasch's theorem and as each of $\overrightarrow{AB}$, $\overrightarrow{AC}$ have a point in common with $\overrightarrow{CD}$ then AF will have a point in common with $\overrightarrow{CD}$. (Cor. to Pasch's theorem which will contradict statement 1)
(4) F is not opposite to C-side of B	(4) If F is opposite to C-side of B then F and A will lie in the opposite half planes when $\overleftrightarrow{CD}$ is taken to be the edge and in that case $\overleftrightarrow{AF}$ will have a point in common with $\overleftrightarrow{CD}$ (Plane separation Postulate) and this will contradict statement 1
(5) F—C—B	(5) Statements, 3 and 4
(6) $\frac{DA}{BD}=\frac{CF}{BC}$	(6) $\overleftrightarrow{CD} \parallel \overleftrightarrow{AE}$. Theorem XII (7)
(7) $\angle DCA \cong \angle CAF$	(7) Alternate interior angles
(8) $\angle BCD \cong \angle CFA$	(8) Corresponding angles

(9) $\angle DCA \cong \angle BCD$	(9) Given
(10) $\angle CAF \cong \angle CFA$	(10) Statements 7, 8, 9
(11) $\overline{CF} \cong \overline{CA}$	(11) Statement 10
(12) $\dfrac{DA}{BD} = \dfrac{CA}{BC}$	(12) Statements 6 and 11.

Theorem XII(**10***a*). The bisector of an angle exterior to angle $\angle BAC$ of a $\triangle ABC$ meets the line $\overleftrightarrow{BC}$ in a point D such that $D-B-C$ or $B-C-D$ and such that $\dfrac{BD}{CD} = \dfrac{AB}{AC}$, the $\triangle ABC$ not having $\angle B$ and $\angle C$ congruent to each other.

Given : A $\triangle ABC$ with $m\angle B \neq m\angle C$. $\angle EAC$ is exterior to $\angle BAC$ and the angle bisector of $\angle EAC$ meets $\overleftrightarrow{BC}$ in D such that $B-C-D$.

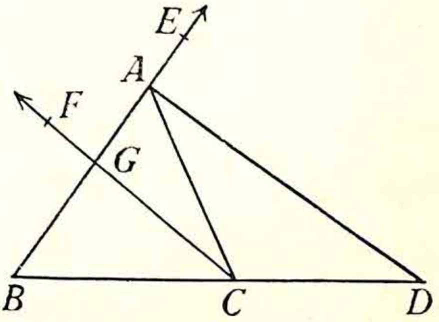


Fig. 197.

To prove that : $\dfrac{BD}{CD} = \dfrac{AB}{AC}$.

Proof.

Statement	*Reason*
(1) Through C, there is a line $\overleftrightarrow{CF} \parallel \overleftrightarrow{DA}$	(1) Parallel postulate
(2) $\overrightarrow{CF}$ has a point G in comwith $\overline{AB}$	(2) $\overrightarrow{CF} \nparallel \overline{AB}$ as $\overrightarrow{CF} \parallel \overline{AD}$ and $\overline{AB}$ and $\overline{AD}$ have a point A in common and if $\overrightarrow{CF} \parallel \overline{AB}$ then it will lead to a contradiction of parallel postulate

(3) G is not opposite to B-side of A (It is not B—A—G)	(3) If B—A—G then $\overrightarrow{CF}$ will intersect $\overline{AD}$ (Plane separation postulate when $\overleftrightarrow{AD}$ is taken as edge)
(4) G is not opposite to A-side of B. (It is not G—B—A)	(4) If G—B—A then $\overrightarrow{CG}$ will not be between $\overrightarrow{CA}$ and $\overrightarrow{CB}$. Consequently $\overrightarrow{GC}$ will be between $\overrightarrow{CA}$ and $\overrightarrow{CD}$ and then $\overrightarrow{GC}$ will have a point in common with $\overline{AD}$ (Cor. to Pasch's theorem) leading to a contradiction of statement 1
(5) B—G—A	(5) Statements 3 and 4
(6) $\frac{CD}{BC}=\frac{GA}{BG}$	(6) Theorem XII (7). $\overleftrightarrow{CG} \parallel \overleftrightarrow{AD}$
(7) $\frac{BC}{CD}=\frac{BG}{GA}$	(7) Statement 6. Invertendo
(8) $\frac{BC+CD}{CD}=\frac{BG+GA}{GA}$	(8) Statement 7 Componendo
(9) $\frac{BD}{CD}=\frac{AB}{GA}$	(9) Definition of betweenness
(10) $\angle DAC \cong \angle ACG$	(10) Interior alternate angles
(11) $\angle EAD \cong \angle AGC$	(11) Corresponding angles
(12) $\angle EAD \cong \angle DAC$	(12) Given
(13) $\angle AGC \cong \angle ACG$	(13) Statements 10, 11 and 12.
(14) $\overline{AG} \cong \overline{AC}$	(14) Statement 13
(15) $\frac{BD}{CD}=\frac{AB}{AC}$	(15) Statements 9 and 15.

In this case, the point D is said to be the point of external division of the line segment $\overline{AB}$. A line is said to be divided harmonically if it is divided internally and externally in the same ratio. We have seen that the angle bisector of an angle of a triangle and the angle bisector of the exterior angle to the same angle divide the line opposite to that angle harmonically.

Similarity between two triangles. Supposing we know that there exists 1—1 correspondence ABC↔DEF between the vertices of two triangles, $\triangle ABC$ ond $\triangle DEF$ and supposing we are given that :

(1) $\angle A \cong \angle D$; $\angle B \cong \angle E$ and $\angle C \cong \angle F$ and also

(2) $\frac{DE}{AB} = \frac{EF}{BC} = \frac{DF}{AC}$, then we say that 1—1 correspondence ABC↔DEF between the vertices of the triangles, $\triangle ABC$ and $\triangle DEF$ is a similarity and we write that '$\triangle ABC \sim \triangle DEF$' which we read as 'The $\triangle ABC$ is similar to $\triangle DEF$'.

There is a convention that $m(\overline{AB})$ where $\overline{AB}$ is opposite the vertex C of a $\triangle ABC$ is denoted by c. Similarly $m(\overline{BC})$ will be denoted by a and $m(\overline{AC})$ will be denoted by b. In other words AB is denoted by c, BC by a and AC by b. Then

we can write $\frac{DE}{AB} = \frac{EF}{BC} = \frac{DF}{AC}$ as $\frac{f}{c} = \frac{d}{a} = \frac{e}{b}$ and then we can further write it as $c, a, b \sim f, d, e$. $a, b, c \sim d, e, f$ would therefore imply that $\frac{d}{a} = \frac{e}{b} = \frac{f}{c}$.

We summarise it by saying that $\triangle ABC \sim \triangle DEF$ if :

(1) There exists 1—1 correspondence between their vertices.

(2) Pairs of corresponding angles are congruent.

(3) Corresponding sides are proportional.

A triangle will be similar to itself if we can set up a correspondence which satisfies the above conditions. The 1—1 correspondence ABC↔ABC between the vertices of a $\triangle ABC$ and itself is a similarity.

We have already talked about 1—1 correspondence between the vertices of $\triangle ABC$ and $\triangle DEF$ when it is a congruence and when we say that $\triangle ABC \cong \triangle DEF$. When this is so then we know without any reference to any set of figures that $\angle A \cong \angle D$; $\angle B \cong \angle E$ and $\angle C \cong \angle F$ and also $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ and $\overline{AC} \cong \overline{DF}$. So the congruence relationship between two triangles is also a similarity relationship as in the case of congruence relationship $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = 1$ or in other words in this case the 'constant of proportionality' is one. So if $\triangle ABC \cong \triangle DEF$ then $\triangle ABC \sim \triangle DEF$.

Moreover whenever we are given that 1—1 correspondence ABC↔DEF between the vertices of two triangles is a similarity and $\triangle ABC \sim \triangle DEF$ then (without any reference to any set of figures) we write that

(1) $\angle A \cong \angle D$; $\angle B \cong \angle E$ and $\angle C \cong \angle F$ and

(2) $a, b, c \sim d, e, f$

We define similarity relationship between two triangles as follows :

Definition XII (4). A one-to-one correspondence between the vertices of two triangles (or between the vertices of any triangle and itself) such that the corresponding angles are congruent and the length of corresponding sides are proportional is called a similarity and the triangles are said to be similar.

The proportionality constant between the lengths of corresponding sides of two triangles may be less than 1 or more than 1 or even equal to 1. But when the proportionality constant equals 1, then the similarity relationship is also congruence relationship. [It may however be noted that if $\triangle ABC \cong \triangle \angle DEF$ then $\triangle ABC \sim \triangle DEF$ but the converse will not always be true. If $\triangle ABC \sim \triangle DEF$ then $\triangle ABC$ will not necessarily be congruent to $\triangle DEF$]. We can extend the concept of similarity to convex polygons of n sides. A triangle is a polygon of three sides. We can suitablly modify the definition of similarity of two triangles to define similarity relation between two polygons of n sides each. Moreover, as in the case of a triangle when 1—1 correspondence ABC→ABC is a similarity or in other words the identity correspondence between the vertices of two triangles is a similarity, the identity correspondence between the vertices of a convex polygon and itself is a similarity. The ratio between the lengths of corresponding sides of two similar polygons is called ratio of simlitude.

It can be verified that similarity relation on the convex polygons of n sides is an equivalence relation as it is reflexive, symmetric as well as transitive. This holds for similarity relation between two triangles as a triangle is a convex polygon of three sides. Now we prove below some theorems.

Theorem XII (11). If one triangle is similar to second under a similarity with proportionality constant k then the second triangle is similar to first with the proportionality constant $\frac{1}{k}$.

Given : 1—1 correspondence ABC↔DEF between the vertices of $\triangle ABC$ and $\triangle DEF$ such that $\triangle ABC \sim \triangle DEF$. In other words we also know that $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\angle C \cong \angle F$ and

$$\frac{DE}{AB} = \frac{EF}{BC} = \frac{DF}{AC} = k$$

To prove that : $\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}=\frac{1}{k}$

Proof.

Statement	*Reason*
(1) $\frac{DE}{AB}=k$	(1) Given
(2) $DE=k.AB$	(2) Statement 1
(3) $\frac{DE}{k.DE}=\frac{k.AB}{k.DE}$	(3) If $a=b$ then $\frac{a}{c}=\frac{b}{c}$ $(c\neq 0)$ (Division Axiom)
(4) $\frac{1}{k}=\frac{AB}{DE}$	(4) Statement 3.

Theorem XII (12). If one triangle is similar to second triangle with proportionality constant k and if the second is similar to the third with the proportionality constant l then the first is similar to the third with the proportionality constant kl. Hence the triangles which are similar to the same triangle are similar to each other (or similarity relationship is transitive).

Given : (1) 1—1 correspondence $ABC\leftrightarrow DEF$ between the vertices of $\triangle ABC$ and $\triangle DEF$ and $\triangle ABC\sim\triangle DEF$ which implies that

$$\frac{DE}{AB}=\frac{EF}{BC}=\frac{DF}{AC}=k.$$

(2) 1—1 correspondence $DEF\leftrightarrow XYZ$ between the vertices of $\triangle DEF$ and $\triangle XYZ$ which implies that

$$\frac{XY}{DE}=\frac{YZ}{EF}=\frac{XZ}{DF}=l.$$

To prove that :

$$\frac{XY}{AB}=\frac{YZ}{BC}=\frac{XZ}{AC}=kl.$$

and $\triangle ABC\sim\triangle XYZ$.

Proof.

Statement	*Reason*
(1) $\frac{DE}{AB}=k$ and $\frac{XY}{DE}=l$	(1) Given
(2) $\frac{DE}{AB}\cdot\frac{XY}{DE}=kl$	(2) Statement 1

(3) $\frac{XY}{AB}=kl$	(3) Statement 2
(4) $\frac{XY}{AB}=\frac{YZ}{BC}=\frac{XZ}{AC}=kl$	(4) As for statement 3.

Also as $\angle A \cong \angle D$ and $\angle D \cong \angle X$, therefore $\angle A \cong \angle X$.

Similarly $\angle B \cong \angle Y$ and $\angle C \cong \angle Z$. Hence $1-1$ correspondence $ABC \leftrightarrow XYZ$ is a similarity and $\triangle ABC \sim \triangle XYZ$.

Theorem XII (13). Given a correspondence between two triangles, if corresponding angles are congruent then the correspondence is a similarity (A.A.A. similarity theorem).

Given : $1-1$ correspondence $ABC \leftrightarrow DEF$ between the vertices of the $\triangle ABC$ and $\triangle DEF$ and $\angle A \cong \angle D$; $\angle B \cong \angle E$ and $\angle C \cong F \angle$.

To prove that : 1—1 correspondence $ABC \leftrightarrow DEF$ is a similarity and $\triangle ABC \sim \triangle DEF$.

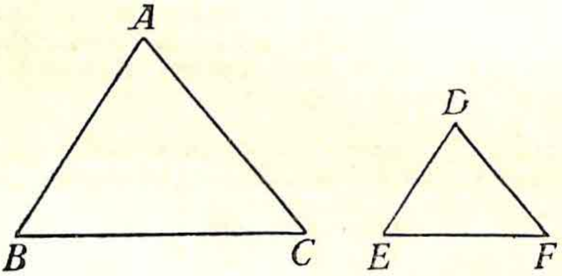


Fig. 198.

Proof. If we consider the measure of the side $\overline{AB}$ of $\triangle ABC$ and the side $\overline{DE}$ of $\triangle DEF$ then (By Trichotomy Law) either :

(1) $AB=DE$ or

(2) $AB>DE$ or

(3) $AB<DE$.

(It may be noted that AB, DE are real numbers which are measures of sides $\overline{AB}$ and $\overline{DE}$ respectively).

First Case. If $AB=DE$ then.

Statement	*Reason*
(1) $\overline{AB} \cong \overline{DE}$	(1) Assumption
(2) $\angle A \cong \angle D$ and $\angle B \cong \angle E$	(2) Given
(3) $1-1$ correspondence $ABC \leftrightarrow DEF$ is a congruence and $\triangle ABC \cong \triangle DEF$	(3) A.S.A. theorem

(4) $\frac{DE}{AB}=\frac{EF}{BC}=\frac{DF}{AC}=1$	(4) Statement 3
(5) 1—1 correspondence ABC↔DEF is a similarity and $\triangle ABC \sim \triangle DEF$.	(6) Statement 4.

Second Case : If AB>DE then :—

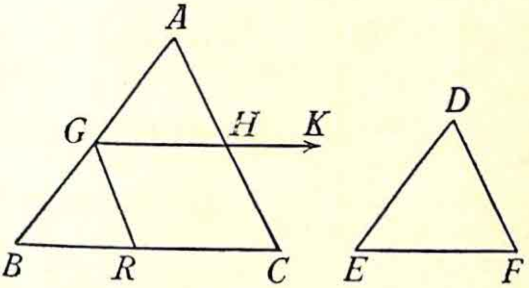


Fig. 199.

Statement	*Reason*
(1) There exists a point G on $\overline{AB}$ such that $\overline{AG} \cong \overline{DE}$ and A—G—B	(1) Point Plotting theorem
(2) Through G, there is a line $\overleftrightarrow{GK} \parallel \overline{BC}$	(2) Parallel Postulate
(3) $\overleftrightarrow{GK}$ meets $\overline{AC}$ in H such that A—H—C	(3) Pasch's theorem
(4) $\angle AGH \cong \angle B$	(4) Alternate interior angles
(5) $\angle B \cong \angle E$	(5) Given
(6) $\triangle AGH \cong \angle E$	(6) Statements 4 and 5
(7) $\angle AHG \cong \angle F$	(7) Reasoning same as for statement 6
(8) 1—1 correspondence AGH ↔DEF is a congruence and $\triangle AGH \cong \triangle DEF$	(8) SAA theorem
(9) $\overline{GH} \cong \overline{EF}$ and $\overline{AH} \cong \overline{DF}$	(9) Corresponding parts of congruent triangles
(10) $\frac{AG}{AB}=\frac{AH}{AC}$	(10) Theorem XII (8)

(11) $\overline{GR}\parallel\overline{HC}$ such that $B-R-C$	(11) Reasoning similar to the one for statements 2 and 3
(12) $\overline{RC}\cong\overline{GH}$	(12) GRCH is a $\parallel^{gm}$. statements 2 and 11
(13) $\frac{AG}{AB}=\frac{RC}{BC}$	(13) Theorem XII (8)
(14) $\frac{AG}{AB}=\frac{AH}{AC}=\frac{RC}{BC}$	(14) Statements 10 and 13
(15) $\frac{DE}{AB}=\frac{DF}{AC}=\frac{EF}{BC}$	(15) Statements 1, 9, 12 and 14.
(16) $\angle A\cong\angle D$; $\angle B\cong\angle E$; $\angle C\cong\angle F$	(16) Given
(17) 1—1 correspondence $ABC\leftrightarrow DEF$ is a similarity and $\triangle ABC\sim\triangle DEF$.	(17) Statements 15 and 16.

Third Case : If $AB<DE$ then there is a point G on DE such that $DG=AB$ and $D-G-E$. Then we can proceed exactly on the lines as for case II and prove the desired result.

Theorem XII (14). (A.A similarity theorem). Given a correspondence between two triangles. If two pairs of corresponding angles are congruent then the correspondence is a similarity.

Proof. We know that sum of angles of a triangle is 180. Hence if two pairs of corresponding angles are congruent then the third pair is also congruent. Hence A.A similarity theorem is same as A.A.A similarity theorem.

Theorem XII (15). (S.S.S. similarity theorem). Given 1-1 correspondence between the vertices of two triangles. If corresponding sides of two triangles are proportional, then their corresponding angles are congruent and the correspondence is a similarity.

Given : $1-1$ correspondence $ABC\leftrightarrow DEF$ between the vertices of $\triangle ABC$ and $\triangle DEF$ such that $a, b, c\sim d, e, f$.

To prove that : $\triangle ABC\sim\triangle DEF$.

Proof. If we consider the measure of side $\overline{AB}$ of $\triangle ABC$ and that of its corresponding side $\overline{DE}$ of $\triangle DEF$ then either

(*i*) $AB=DE$ or (*ii*) $AB>DE$ or (*iii*) $AB<DE$.

1st Case. If $AB=DE$ then $\frac{DE}{AB}=1$ or $\frac{f}{c}=1$ and as we are given that $a, b, c\sim d, e, f$ it implies that $\frac{d}{a}=\frac{e}{b}=\frac{f}{c}=1$ or it implies that $d=a$ and $e=b$ or $\overline{EF}\cong\overline{BC}$ and $\overline{DF}\cong\overline{AC}$. Then :

Statement	*Reason*
(1) 1−1 correspondence $ABC \leftrightarrow DEF$ is a congruence $\triangle ABC \cong \triangle DEF$	(1) S.S.S. Theorem
(2) $\angle A \cong \angle D$; $\angle B \cong \angle E$ and $\angle C \cong \angle F$	(2) Corresponding parts of congruent triangles.
(3) $a, b, c \sim d, e, f$	(3) Given
(4) 1−1 correspondence $ABC \leftrightarrow DEF$ is a similarity and $\triangle ABC \sim \triangle DEF$.	(4) Statements 2 and 3.

Second Case. If $AB > DE$ then :

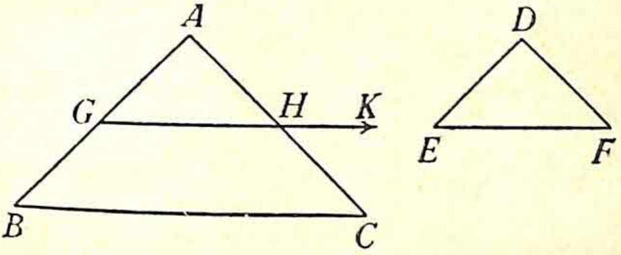


Fig. 200.

Statement	*Reason*
(1) There is a point G on $\overline{AB}$ such that A−G−B and $\overline{AG} \cong \overline{DE}$	(1) Point Plotting Theorem
(2) Through G there exists a ray $\overrightarrow{GK} \parallel \overrightarrow{BC}$	(2) Parallel Postulate
(3) $\overrightarrow{GK}$ does not pass through A or C	(3) If $\overrightarrow{GK}$ passes through A or C then it will contradict the parallel postulate
(4) $\overrightarrow{GK}$ contains a point H which is an interior point of $\overline{AC}$	(4) Pasch's theorem
(5) $\angle AGH \cong \angle B$	(5) Corresponding angles
(6) $\angle AHG \cong \angle C$	(6) Corresponding angles

(7) $\angle GAH \cong \angle A$	(7) An angle is congruent to itself
(8) 1−1 correspondence AGH↔ABC is a similarity and $\triangle AGH \sim \triangle ABC$	(8) A.A.A. similarity theorem
(9) AG, GH, AH$\sim c, a, b$	(9) Statement 8
(10) $\frac{c}{AG}=\frac{a}{GH}=\frac{b}{AH}$	(10) Statement 9
(11) $\frac{c}{f}=\frac{a}{GH}=\frac{b}{AH}$	(11) AG=DE and DE=f
(12) $a, b, c \sim d, e, f$	(12) Given
(13) $\frac{d}{a}=\frac{e}{b}=\frac{f}{c}$	(13) Statement 12
(14) $\frac{c}{f}=\frac{b}{e}=\frac{a}{d}$	(14) Statement 13 (Invertendo)
(15) $\frac{c}{f}=\frac{b}{e}=\frac{b}{AH}$	(15) Statements 11 and 14
(16) AH=e	(16) Statement 15
(17) $\frac{c}{f}=\frac{a}{d}=\frac{a}{AH}$	(17) Statements 11 and 14
(18) GH=d	(18) Statement 17
(19) $\overline{GH} \cong \overline{EF}$, $\overline{AH} \cong \overline{DF}$ and $\overline{AG} \cong \overline{DE}$	(19) Statements 16 and 18
(20) 1−1 correspondence AGH↔DEF is a congruence and $\triangle AGH \cong \triangle DEF$	(20) S.S.S. Theorem
(21) $\angle G \cong \angle E$, $\angle H \cong \angle F$ and $\angle A \cong \angle D$	(21) Corresponding parts of congruent triangles
(22) $\angle B \cong \angle E$ and $\angle C \cong \angle F$ and $\angle A \cong \angle D$	(22) Why ? Statements 5, 6, 7 and 21
(23) 1−1 correspondence ABC↔DEF is a similarity and $\triangle ABC \sim \triangle DEF$.	(23) A.A.A. similarity theorem.

Third Case. If AB<DE. This can be proved on similar lines as case II. How ?

Theorem XII (16). (*S.A.S. Similarity theorem*). Given two triangles and correspondence between their vertices. If two pairs of corresponding sides are proportional and the included angles are congruent then the correspondence is a similarity.

Given : $1-1$ correspondence $ABC \leftrightarrow DEF$ between the vertices of $\triangle ABC$ and $\triangle DEF$ such that $b, c \sim e, f$ and $\angle A \cong \angle D$.

To Prove that : $\triangle ABC \sim \triangle DEF$.

Proof. We are given that $b, c \sim e, f$ or $\frac{e}{b} = \frac{f}{c}$

Either $\frac{e}{b} = 1$ or $\frac{e}{b} < 1$ or $\frac{e}{b} > 1$ (Trichotomy law)

First Case : $\frac{e}{b} = 1$ or $\frac{DF}{AC} = 1$ or $\overline{DF} \cong \overline{AC}$

and also $\frac{f}{c} = \frac{e}{b} = 1$ or $f = c$ or $\overline{DE} \cong \overline{AB}$, also $\angle A \cong \angle D$ (given).

Therefore $1-1$ correspondence $ABC \leftrightarrow DEF$ is a congruence and $\triangle ABC \leftrightarrow \triangle DEF$. It emplies that $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\angle C \cong \angle F$.

Hence $1-1$ correspondence $ABC \leftrightarrow DEF$ is a similarity and $\triangle ABC \sim \triangle DEF$ (A.A.A. similarity theorem). The reader may write this proof in columns.

Second case : If $\frac{e}{b} < 1$ or $\frac{DF}{AC} < 1$ then $m(\overline{DF}) < m\,\overline{AC}$ or $m\,\overline{AC} > m\,\overline{DF}$ then :

Statement	*Reason*
(1) There is a point G in $\overline{AC}$ such that A—G—C and $\overline{AG} \cong \overline{DF}$	(1) Point Plotting theorem

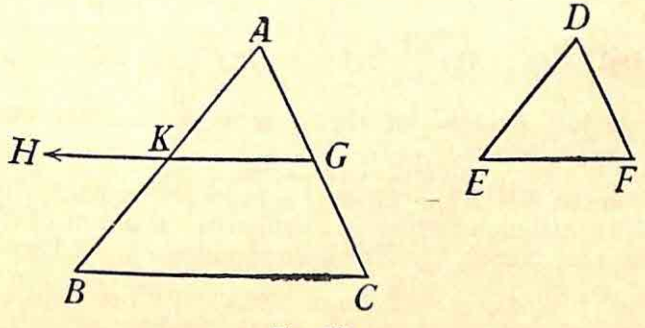


Fig. 201

(2) $\overrightarrow{GH} \parallel \overline{CB}$	(2) Parallel Postulate
(3) $\overleftrightarrow{GH}$ has a point K in common with $\overline{AB}$ such that A—K—B	(3) Why ?
(4) 1—1 correspondence AKG ↔ABC is similarity and $\triangle AKG \sim \triangle ABC$	(4) A.A.A similarity theorem. How ?
(5) $\frac{AC}{AG}=\frac{AB}{AK}=\frac{BC}{KG}$	(5) Statement 4
(6) $\frac{AC}{DF}=\frac{AB}{AK}=\frac{BC}{GK}$	(6) $\overline{AG}\cong\overline{DF}$
(7) $\frac{DF}{AC}=\frac{AK}{AB}=\frac{GK}{BC}$	(7) Statement 6
(8) $\frac{DF}{AC}=\frac{DE}{AB}$	(8) Given
(9) $\frac{DF}{AC}=\frac{AK}{AB}=\frac{DE}{AB}$	(9) Statements 7 and 8
(10) $AK=DE$ or $\overline{AK}\cong\overline{DE}$	(10) Statement 9
(11) 1—1 correspondence AKG↔DFF is a congruence and $\triangle AKG \cong \triangle DEF$	(11) S.A.S Postulate
(12) 1—1 correspondence AKG ↔DEF is a similarity and $\triangle AKG \sim \triangle DEF$	(12) Statement 11
(13) $\triangle AKG \sim \triangle ABC$	(13) Statement 4
(14) $\triangle ABC \sim \triangle DEF$	(14) Statements 12 and 13 (How ?)

Third Case : If $\frac{DF}{AC} > 1$ then $m\,\overline{DF} > m\,\overline{AC}$.

Proceeding exactly on the lines as for second case, we can prove the desired result.

Theorem XII (17). Given 1—1 correspondence between the vertices of two triangles which is a similarity. If a pair of corresponding sides are congruent, then the correspondence is a congruence.

Given : 1—1 correspondence ABC↔DEF between the vertices of $\triangle ABC$ and $\triangle DEF$ such that $\triangle ABC \sim \triangle DEF$ and $\overline{AB}\cong\overline{DE}$.

To Prove that : 1—1 correspondence ABC↔DEF is a congruence and $\triangle ABC \cong \triangle DEF$.

Proof.

Statement	*Reason*
(1) $a, b, c \sim d, e, f$ or $\frac{d}{a}=\frac{e}{b}=\frac{f}{c}$	(1) $\triangle ABC \sim \triangle DEF$
(2) $\frac{f}{c}=1$	(2) $\overline{AB} \cong \overline{DE}$ or $c=f$
(3) $\frac{d}{a}=\frac{e}{b}=\frac{f}{c}=1$	(3) Statements 1 and 2
(4) $d=a$ and $e=b$	(4) Statement 3
(5) $\overline{EF} \cong \overline{BC}$ and $\overline{DF} \cong \overline{AC}$	(5) Statement 4
(6) $1-1$ correspondence ABC↔DEF is a congruence and $\triangle ABC \cong \triangle DEF$	(6) S.S.S Theorem.

Theorem XII (18). Given $\triangle ABC$ and a positive number k. Then there exists a triangle which is similar to $\triangle ABC$ with proportionality constant k.

Proof.

First Case : If $k=1$ then $\triangle ABC$ is similar to itself.

Seond Case : If $k<1$ then there exists a point D such that $A-D-B$ and $AD=k \times AB$ (point plotting theorem). Through D there is a line parallel to $\overline{BC}$ intersecting $\overline{AC}$ in E such that $A-E-C$ (Parallel postulate, Pasch's theorem). $\angle D \cong \angle B$ and $\angle E \cong \angle C$ (corresponding angles). Also $\frac{AD}{AB}=\frac{AE}{AC}$ ($\overline{DE} \parallel \overline{BC}$). As $AD=k.AB$, therefore $AE=k.AC$.

Through D, there is a line parallel to $\overline{EC}$ meeting $\overline{BC}$ in F such that $B-F-C$. Therefore $\frac{AB}{AD}=\frac{BC}{FC}$ (why ?). As $AD=k.AB$, therefore $FC=k.BC$. Also DFCE is a $\parallel^{gm}$ (How ?). Therefore $\overline{FC} \cong \overline{DE}$ and hence $DE=k.BC$. Hence $\frac{AD}{AB}=\frac{DE}{BC}=\frac{AE}{AC}=k$. Also in $\triangle ADE$ and $\triangle ABC$,

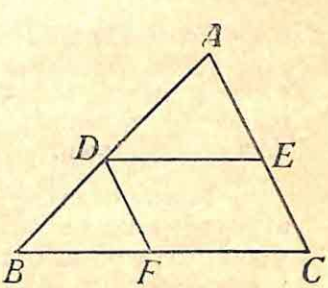


Fig. 202.

corresponding angles are congruent. Hence 1–1 correspondence ABC↔DEF is a similarity and $\triangle ABC \sim \triangle DEF$.

Third Case. If $k>1$, then there is a point D on $\overleftrightarrow{AB}$ such that A–B–D and $AD=k.AB$. A line through D parallel to $\overline{BC}$ intersects $\overleftrightarrow{AC}$ in a point E such that A–C–E (How ?) Also $\frac{AD}{AB}=\frac{AE}{AC}$. Hence $AE=k.AC$. Through B, there is a line parallel to $\overleftrightarrow{CE}$ meeting $\overline{DE}$ in F such that D–F–C (How ?) and $\frac{AD}{AB}=\frac{DE}{FE}$ and as $AD=k.AB$, therefore $DE=k.FE$. But as BFEC is a $\parallel^{gm}$, therefore $FE=BC$. Hence $DE=k.BC$. Also in $\triangle ABC$ and $\triangle ADE$, the corresponding angles are congruent. So $\triangle ADE \sim \triangle ABC$. Hence in both these cases $\triangle ADE$ is similar to $\triangle ABC$ with a proportionality constant k.

We could make use of this theorem in proving S.S.S similarity theorem, S.A.S. similarity theorem and A.A.A similarity theorem. In proving the above theorem we have not made use of any of these.

Let us prove S.S.S similarity theorem with its help.

Let ABC↔DEF be 1–1 correspondence between the vertices of $\triangle ABC$ and $\triangle DEF$. Also we are given that

$$\frac{DE}{AB}=\frac{EF}{BC}=\frac{DF}{AC}=k \text{ (say)} \qquad \ldots(1)$$

Proof. There exists a $\triangle XYZ$ similar to $\triangle DEF$ with proportionality constant k. Hence :

$$\frac{DE}{XY}=\frac{EF}{YZ}=\frac{DF}{XZ}=k \qquad \ldots(2)$$

$$\text{and } \angle X \cong \angle D\,;\ \angle Y \cong \angle E \text{ and } \angle Z \cong \angle F \qquad \ldots(3)$$

From (1) and (2) we get $\overline{AB}\cong\overline{XY}$, $\overline{BC}\cong\overline{YZ}$ and $\overline{AC}\cong\overline{XZ}$. Hence $\triangle ABC \cong \triangle XYZ$ with 1–1 correspondence ABC↔XYZ being a congruence. (S.S.S. Theorem). Thus $\angle A\cong\angle X$, $\angle B\cong\angle Y$ and $\angle C\cong\angle Z$. ...(4)

From (3) and (4) we get that $\angle A\cong\angle D$, $\angle B\cong\angle E$ and $\angle C\cong\angle F$. As we are given that $a, b, c \sim d, e, f$, therefore 1–1 correspondence ABC↔DEF is a similarity and $\triangle ABC \sim \triangle DEF$. The reader may prove S.A.S. and A.A.A. similarity theorems on these lines on their own.

In a right triangle, the side opposite the right angle is known as hypotenuse and other two sides are known as legs. Also we know that altitude of a triangle is the perpendicular segment from the vertex to the line containing the opposite side. We have also proved that the foot of perpendicular (altitude) from the vertex of the right angle of a right triangle to the opposite side (hypotenuse) lies in the interior of that side. Keeping these things in view we prove the following theorems :

Theorem XII (19). In any right triangle, the altitude to the hypotenuse forms with the respective legs and corresponding segments of the hypotenuse, two triangles which are similar to each other and to the original triangle.

Given : A $\triangle ABC$. $m\angle A=90$. D is the foot of the perpendicular from A to $\overline{BC}$.

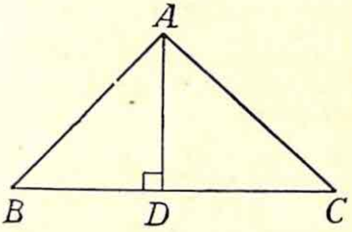


Fig. 203.

To prove that :

(1) $1-1$ correspondence $ABD \leftrightarrow CBA$ is a similarity and $\triangle ABD \sim \triangle CBA$.

(2) $1-1$ correspondence $ACD \leftrightarrow BCA$ is a similarity and $\triangle ACD \sim \triangle BCA$.

(3) $1-1$ correspondence $ABD \leftrightarrow CAD$ is a similarity and $\triangle ABD \sim \triangle CAD$.

Proof. (First Part) B—D—C. Why ?

Statement	*Reason*
(1) $\angle ABD \cong \angle B$	(1) An angle is congruent to itself
(2) $\angle ADB \cong \angle BAC$	(2) Each is a right angle
(3) $1-1$ correspondence $ABD \leftrightarrow CBA$ is a similarity and $\triangle ABD \sim \triangle CBA$	(3) A.A. similarity theorem

Second Part. The proof is similar to Part I

Third Part :

Statement	*Reason*
(1) $\angle ABD \cong \angle DAC$	(1) Measure of each is 90—measure of $\angle C$

(2) $\angle ADB \cong \angle ADC$	(2) Each is a right angle
(3) $1-1$ correspondence $ABD \leftrightarrow CAD$ is a similarity and $\triangle ABD \sim \triangle CAD$.	(3) A.A similarity theorem

Theorem XII (20). If there is $1-1$ correspondence between the vertices of two triangles which is a similarity with proportionality constant k then the lengths of altitudes of first triangle are proportional to the lengths of corresponding altitudes of the second triangle with the proportionality constant k.

Given : $1-1$ correspondence $ABC \leftrightarrow DEF$ between the vertices of $\triangle ABC$ and $\triangle DEF$ and $a, b, c \sim d, e, f$ with proportionality constant k. In other words

$$\frac{d}{a}=\frac{e}{b}=\frac{f}{c}=k$$

or $\quad d=ak, \quad e=bk, \quad f=ck,$

or $\quad EF=k.BC, \quad DF=k.AC$

and $\quad DE=k.AB.$

To prove that : If $\overline{AG}$ is the altitude from A to $\overline{BC}$ of $\triangle ABC$ and $\overline{DH}$ is the altitude from D to $\overline{EF}$ of the $\triangle DEF$ then $DH=k.AG$ and two other similar results.

Proof. If we consider the measures of $\angle C$ of $\triangle ABC$ and $\angle F$ of $\triangle DEF$, then there are three alternatives.

(1) Each of $\angle C$ and $\angle F$ is a right angle.

(2) Each of $\angle C$ and $\angle F$ is acute.

(3) Each of $\angle C$ and $\angle F$ is obtuse.

First Case.

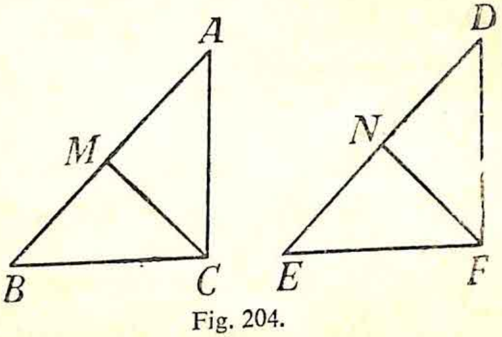


Fig. 204.

When each of $\angle C$ and $\angle F$ is a right angle then the altitude $\overline{AG}=\overline{AC}$ and altitude $\overline{DH}=\overline{DF}$. As we are given that $DF=k.AC$

therefore $DH=k.AG$. Similarly if $\overline{BK}$ is the altitude from B to $\overline{AC}$ and $\overline{EL}$ is the altitude from E to $\overline{DF}$ then $\overline{BK}=\overline{BC}$ and $\overline{EL}=\overline{EF}$ and hence we prove that $EL=k.BK$. To prove the third result if $\overline{CM}\perp\overline{AB}$ and $\overline{FN}\perp\overline{DE}$ then 1−1 correspondence CMA↔FND is a similarity and $\triangle CMA\sim\triangle FND$ as $\angle A\cong\angle D$ and $\angle CMA\cong\angle FND$, each being right angle. (A.A. similarity theorem). Hence

$$\frac{FN}{CM}=\frac{FD}{CA}=k$$

or

$$FN=k.CM$$

Second Case :

When each of $\angle C$, $\angle F$ is acute, then B−G−C and E−H−F (Why ?)

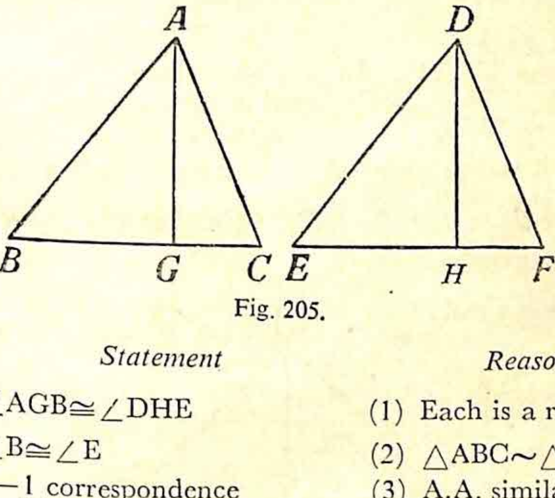


Fig. 205.

Statement	*Reason*
(1) $\angle AGB\cong\angle DHE$	(1) Each is a right angle
(2) $\angle B\cong\angle E$	(2) $\triangle ABC\sim\triangle DEF$
(3) 1−1 correspondence ABG↔DEH is a similarity and $\triangle ABG\sim\triangle DEH$	(3) A.A. similarity
(4) $DH=k.AG$	(4) Statement 3 and as $DE=k.AB$ (given).

Similarly we can prove the other two results.

Third Case. When each of $\angle C$ and $\angle F$ is obtuse, then B−C−G and E−F−H. We can proceed on similar lines as for the second case and prove the three results.

The Pythagorean Theorem. We can make use of the concept of similarity between two triangles in proving the most famous and one of the most important theorems of Geometry—The theorem of Pythagoras (580 ?−500 ? B.C.). Pythagoras is said to have discovered the theorem that the square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides. It is said that the Egyptians were the first to discover the truth of this theorem in the special case when the sides of a triangle are 3, 4, 5 respectively. It is said that Pythagoras was so happy over the discovery of this theorem that he sacrificed a hecatomb. Some Historians doubt the correctness of this statement as they say that Pythagoras did not believe in shedding the blood as they believed in transmigration of soul. They suggest that he might have sacrificed 'an OX made of flour'.

The proof of the theorem which is based on the concept of similarity that we are going to discuss is due to the Indian Mathematician Bhaskra. His proof depends on drawing from the vertex of right triangle, a perpendicular to hypotenuse and comparing the two triangles thus obtained with the given triangle to which they are similar. This proof of the theorem was unknown to Europe till John Wallis (1606—1703) rediscovered it. This proof is given below.

Theorem XII (21). In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of its two legs.

Given : A right triangle ABC with right angle at C.

To prove that : $(AB)^2=(AC)^2+(BC)^2$ or $c^2=b^2+a^2$.

Proof.

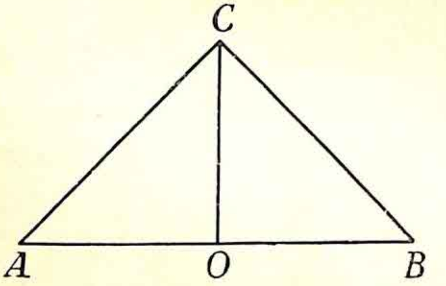


Fig. 206.

Statement	*Reason*
(1) O is the foot of altitude from C to AB and A–O–B	(1) Why A–O–B ? Refer Problem D-8, Chapter IX
(2) 1—1 correspondence ACO ↔ABC is a similarity and $\triangle ACO \sim \triangle ABC$	(2) Theorem XII (19)
(3) 1—1 correspondence CBO↔ABC is a similarity and $\triangle CBO \sim \triangle ABC$	(3) Theorem XII (19)

(4) $\frac{AC}{AB}=\frac{CO}{BC}=\frac{AO}{AC}=k$	(4) Statement 2
(5) $\frac{CB}{AB}=\frac{BO}{BC}=\frac{CO}{AC}=l$	(5) Statement 3
(6) $AO=k.\,AC$ or $AO=k.b$	(6) Statement 4
(7) $OB=l.BC$ or $OB=l.a$	(7) Statement 5
(8) $AO+OB=AB$	(8) A—O—B. Definition of betweenness
(9) $kb+la=c$	(9) Statements 6, 7, 8
(10) $k=\frac{AC}{AB}=\frac{b}{c}$	(10) Statement 4
(11) $l=\frac{CB}{AB}=\frac{a}{c}$	(11) Statement 5
(12) $\frac{b}{c}\,.\,b+\frac{a}{c}\,.\,a=c$	(12) Statements 9, 10, 11
(13) $\frac{b^2}{c}+\frac{a^2}{c}=c$	(13) Statement 12
(14) $b^2+a^2=c^2$	(14) Statement 13.

Theorem XII (22). (*Converse to Pythagorean's theorem*). If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of other two sides, the triangle is a right triangle with the right angle opposite the first or the longest side.

Given : A $\triangle ABC$ and $c^2=a^2+b^2$.

To prove that : $\angle C$ is a right angle.

Proof. Consider two perpendicular rays $\overrightarrow{DE}$ and $\overrightarrow{DF}$.

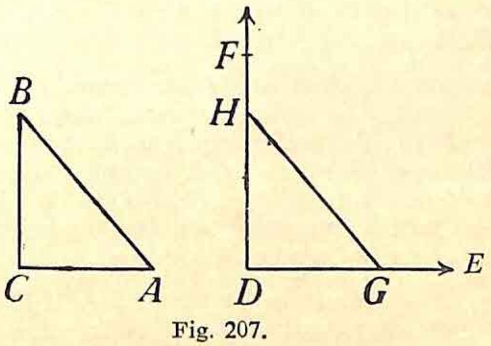


Fig. 207.

Statement	*Reason*
(1) On $\overrightarrow{DE}$, there is a point G such that DG$=b$ and on $\overrightarrow{DF}$ there is a point H such that DH$=a$	(1) Segment construction Theorem
(2) $(HG)^2=a^2+b^2$	(2) DHG is a right triangle. Hence $(HG)^2=(DG)^2+(DH)^2$. Theorem XII (21)
(3) $c^2=a^2+b^2$	(3) Given
(4) $c^2=(HG)^2$	(4) Statements 2 and 3
(5) $c=HG$	(5) Statement 4
(6) 1—1 correspondence BCA $\leftrightarrow$HDG is a congruence and $\triangle BCA\cong\triangle HDG$	(6) S.S.S. theorem. Statements 1 and 5
(7) $\angle C\cong\angle D$	(7) Corresponding parts of congruent triangles
(8) $\angle C$ is a right angle	(8) As $\angle D$ is a right angle.

Pythagorean Triplets. We know that a △ with sides 3, 4, 5 is a right triangle $(3)^2+(4)^2=(5)^2$. (3, 4, 5) is, therefore, known as a Pythagorean triplet. Similarly $(5)^2+(12)^2=(13)^2$, therefore (5, 12, 13) is also a Pythagorean triplet. It is easy to prove that if (a, b, c) is a Pythagorean triplet then (la, lb, lc) will also be a Pythagorean triplet. As (3, 4, 5) is a Pythagorean triplet, therefore (6, 8, 10), (9, 12, 15), (12, 16, 20)...are all Pythagorean triplets.

A Pythagorean triplet like (3, 4, 5) is called a primitive Pythagorean triplet as 3, 4, 5 are prime to each other. In fact 3, 4, 5 are prime to each other in pairs. Now if we can find a formula that can help us in giving all primitive Pythagorean triplets, then we can write all Pythagorean triplets if we have the capacity and the time to do so.

All Pythagorean triplets satisfy the formula $a^2+b^2=c^2$. Now if (a, b, c) is to be a primitive Pythagorean triplet then (a, b), (b, c), (a, c) should be prime. Now if a and b are to be prime to each other then a, b will not both be even. a, b cannot both be odd also, as if a, b are both odd then a^2 will be of the form $4q+1$ and b^2 will be of the form $4r+1$ and then a^2+b^2 will be equal to $4q+4r+2$ or $4(q+r)+2$ which cannot be a perfect square.

Theory of Numbers tells us that if (a, b, c) is a primitive Pythagorean triplet, then there exist integers x, y such that x, y are prime to each other and if one of these is odd then the other is even and such that :

(*i*) $a=2xy$ (*ii*) $b=x^2-y^2$ (*iii*) $c=x^2+y^2$. Now $a^2+b^2=(2xy)^2+(x^2-y^2)^2=(x^2+y^2)^2=c^2$. With these equations we can choose a, b, c if we can choose x, y such that if one of these is odd, then the other is even. Let $x=3$ and $y=2$, then $a=12$, $b=5$ and $c=13$; (5, 12, 13) is a primitive Pythogorean triplet.

Problem Set

A

1. Parallel lines intercept proportional segments on two transversals.

$\left(\textbf{Hint}: \frac{BD}{DF}=\frac{BG}{GE} \text{ (why ?) and } \frac{EG}{GB}=\frac{EC}{CA}\right)$

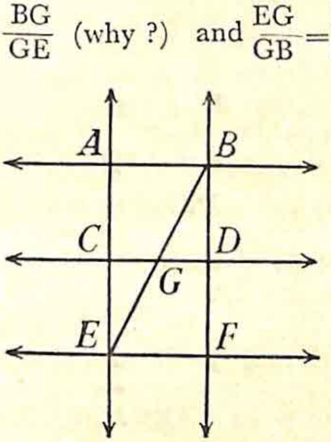


Fig. 208.

2. Prove that the locus of the mid points of line segments drawn through a given point (not in the line) to a given line is a line parallel to a given line.

3. A line through the centroid of a triangle and parallel to one of its sides divides the other two sides into segments having the ratio 2 : 1.

4. Prove that if a line does not divide the two sides of a triangle proportionally, then it is not parallel to the third side.

(**Hint :** If a line l intersects $\overline{AB}$ and $\overline{AC}$ in E & F but does ' ot divide the sides proportionally or $\frac{AE}{EB} \neq \frac{AF}{FB}$, then assume that l is $\parallel$ to $\overline{BC}$. Now find a point G in $\overline{AC}$ such that $\frac{AE}{EB}=\frac{AG}{GB}$, $G \neq F$. Now $\overline{EG} \parallel \overline{BC}$ and $EG \neq l$. This leads to contradiction of parallel postulate. (This theorem is contrapositive of theorem XII (1).

5. If three or more parallel lines intersect segments on two transversals, then the ratio of measures of the segments on one

transversal equals the ratio of measures of the corresponding segments on the other transversal.

6. D is a point on the side $\overrightarrow{AB}$ of $\angle BAC$ such that $A-D-B$. Prove that a line through D parallel to $\overleftrightarrow{BC}$ contains a point E on $\overrightarrow{AC}$ such that $A-E-C$ (use the concept of betweenness and Theorem XII (1).)

7. In a quadrilateral ABCD, $\overline{AD} \parallel \overline{BC}$. There is a point E in $\overline{AB}$ such that $A-E-B$ and a line through E parallel to $\overline{BC}$ meets $\overline{DC}$ in F. Prove that

$$\frac{AE}{EB}=\frac{DF}{FC}.$$

8. P is a point within a quadrilateral ABCD. There are points E, F, G, H on $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, $\overline{PD}$ respectively such that $A-E-P$, $B-F-P$, $C-G-P$ and $D-H-P$. If it is given that $\overline{EF} \parallel \overline{AB}$, $\overline{FG} \parallel \overline{BC}$ and $\overline{GH} \parallel \overline{DC}$ the prove that $\overline{EH} \parallel \overline{AD}$.

9. Given $\triangle ABC$ and $\triangle DEF$ such that $\overline{AB} \parallel \overline{DE}$; $\overline{BC} \parallel \overline{EE}$ and $\overrightarrow{DA}$, $\overrightarrow{EB}$ and $\overrightarrow{FC}$ meet in P such that $D-A-P$, $E-B-P$ and $F-C-P$. Prove that $\overline{AC} \parallel \overline{DF}$.

10. Given $\triangle ABC$ and $\triangle DEF$ such that $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ are parallel to each other in pairs. Also $\overleftrightarrow{AC} \parallel \overleftrightarrow{DF}$. $\overleftrightarrow{BA}$ and $\overleftrightarrow{ED}$ meet in G and $\overleftrightarrow{BC}$ and $\overleftrightarrow{EF}$ meet in H. Prove that $\overleftrightarrow{AC} \parallel \overleftrightarrow{DF} \parallel \overleftrightarrow{GH}$.

11. D, E are the mid points of the sides $\overline{AB}$ and $\overline{AC}$ of $\triangle ABC$. Prove that $\overline{DE} \parallel \overline{BC}$.

12. Given three lines p, q, r. There are points P and $A \in p$, Q and $B \in q$ and R and $C \in r$ such that $\overline{PQ} \parallel \overline{AB}$ and $\overline{QR} \parallel \overline{BC}$. Also P, Q, R are not collinear. Also A, B, C are not collinear. Prove that $\overline{PR} \parallel \overline{AC}$.

13. P is a point in the interior of $\triangle ABC$. $\overline{DE} \parallel \overline{AB}$ where $D \in \overline{AP}$ and $E \in \overline{BP}$. Also $\overline{EF} \parallel \overline{BC}$ where $F \in \overline{CP}$. Prove that $\overline{DF} \parallel \overline{AC}$.

14. A line $\overleftrightarrow{AB}$ divides a plane into two half planes H_1 and H_2. C is a point in H_1 and D is a point in H_2. $E \in \overleftrightarrow{AB}$ such that $A-E-B$. $\overline{EF} \parallel \overline{AC}$ where $F \in BC$ and $\overline{EG} \parallel \overline{AD}$ where $G \in \overline{BD}$. Prove that $\overline{FG} \parallel \overline{CD}$.

15. ABCD is $\parallel^{gm}$. $P \in \overline{AC}$ such shat $A-C-P$. $\overline{BP}$ has a point H in common with $\overleftrightarrow{DC}$ such that $D-C-H$. $\overline{DP}$ has a point K in common with $\overleftrightarrow{BC}$ such that $B-C-K$. Prove that $\overline{HK} \parallel \overline{BD}$.

16. $P \in \overline{BC}$ of a $\triangle ABC$. $\overline{PQ} \parallel \overline{BA}$ where $Q \in \overline{CA}$ and $\overline{PR} \parallel \overline{CA}$ where $R \in \overline{BA}$. $\overleftrightarrow{QR}$ has a point S in common with $\overline{BC}$ such that $S-B-C$. Prove that $(SP)^2 = SB.SC$.

17. $D \in \overline{BC}$ of a $\triangle ABC$ such that $BD = DC$. $E \in \overline{AD}$ such that $AE = ED$. $\overrightarrow{BE}$ has a point X in common with $\overline{AC}$. Prove that $BE : EX = 3 : 1$.

18. Prove that diagonals of a trapezium cut each other proportionally.

B

1. Is it possible that the bisector of an angle exterior to an angle of a triangle be parallel to one side of the triangle? If it so then what kind of the $\triangle$, it would be?

2. Prove that the angle bisector of an angle of a triangle and the angle bisector of the angle exterior to the same angle of the triangle divide the side opposite to the given angle harmonically.

3. $\overline{AD}$ is the median of a $\triangle ABC$. $\overline{DE}$ is the angle bisector of $\angle ADB$ where $E \in AB$ and $\overline{DF}$ is the angle bisector of $\angle ADC$ where $F \in \overline{AC}$. Prove that $\overline{EF} \parallel \overline{BC}$.

$$\left(\textbf{Hint : } \frac{AD}{DB} = \frac{AE}{EB} \text{ and } \frac{AD}{DC} = \frac{AF}{FC} \text{ and } DB = DC \right)$$

4. D is the mid point of the side $\overline{BC}$ of a $\triangle ABC$. $\overline{DE}$ is the angle bisector of $\angle ADB$, $E \in \overline{AB}$. $\overline{EF} \parallel \overline{BC}$ where $F \in \overline{AC}$. Prove that $\angle EDF$ is a right angle.

5. P is any point in the interior of $\triangle ABC$. $\overline{PD}$ is the angle bisector of $\angle BPC$ where $D \in BC$. $\overline{PE}$ is the angle bisector of $\angle APC$ where $E \in \overline{AC}$. $\overline{PF}$ is the angle bisector of $\angle APB$ where $F \in \overline{AB}$. Prove that $BD.\ CE.AF = DC.\ AE.\ BF$.

6. The angle bisectors of $\angle B$ and $\angle C$ of a $\triangle ABC$ intersect the sides $\overline{AC}$ and $\overline{AB}$ in D and E respectively. If $\overline{DE} \parallel \overline{BC}$, prove that $\triangle ABC$ is isosceles.

7. The angle bisectors of $\angle A$ and $\angle C$ of a quadrilateral ABCD have a point F in common such that $F \in \overline{BD}$. Prove that the bisectors of $\angle B$ and $\angle D$ have also the point F in common and also point $F \in \overline{AC}$.

C

1. Prove that if two right triangles have an acute angle of one congruent to an acute angle of the other, the triangles are similar.

2. Two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have a point E in common such

that A−E−B and C−E−D. If $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$, prove that

$$\frac{AE}{BE}=\frac{AC}{BD}.$$

3. $\angle BAC$ is a right angle and $\overline{AD}\perp\overline{BC}$ where $D\in\overline{BC}$. Prove that $\frac{BD}{BA}=\frac{BA}{BC}$.

4. A tower casts a shadow of 30 metres when a 5 metre high telephone pole casts a shadow 4 metres long. How high is the tower ?

5. $\overline{AD}$, $\overline{BE}$ are the altitudes of a $\triangle ABC$ and $\overline{AD}$, $\overline{BE}$ have a point F in common such that B−F−E, A−E−C, and B−D−C. Prove that $\frac{AF}{BF}=\frac{EF}{FD}$ and (*ii*) $\frac{AD}{BE}=\frac{AC}{BC}$.

6. Given a $\triangle ABC$ with B being a right angle. D is a point in $\overline{AC}$ such that A−D−C. $\overrightarrow{DK}\parallel\overrightarrow{CB}$ and $\overrightarrow{DL}\parallel\overrightarrow{AB}$ where $K\in\overline{AB}$ and $L\in\overline{BC}$. A line through B parallel to $\overleftrightarrow{AC}$ has a point E in common with $\overrightarrow{DK}$ and a point F in common with $\overrightarrow{DL}$. Prove that 1−1 correspondence $ABC\leftrightarrow FDE$ is a similarity and $\triangle ABC\sim\triangle FDE$.

7. Prove that the perimetres of similar triangles are proportional to the lengths of any pair of corresponding angles.

8. Prove that equilateral triangles are similar.

9. Prove that two isosceles triangles are similar if the vertex angle of one triangle is congruent to the vertex angle of the other.

10. If two isosceles triangles have a pair of corresponding angles congruent, then the triangles are similar.

11. Given A $\triangle ABC$ such that $\overline{AD}\perp\overline{BC}$ and $\overline{BE}\perp\overline{AC}$ and B−D−C and A−E−C. Prove that 1−1 correspondence $DCA\leftrightarrow ECB$ is a similarity and $\triangle DCA\sim\triangle ECB$.

12. Given $\angle EAC$. A point F on $\overrightarrow{AE}$ such that A−F−E and a point B on $\overrightarrow{AC}$ such that A−B−C. Prove that 1−1 correspondence $EAB\leftrightarrow CAF$ is a similarity and $\triangle EAB\sim\triangle CAF$ if $\angle AEB\cong\angle ACF$. Also prove that $\frac{AC}{AE}=\frac{AF}{AB}$.

13. Given a $\parallel^{gm}$ ABCD and a point E in $\overline{BC}$ such that B—E—C. $\overline{AE}$ and $\overline{BD}$ have a point F in common. Prove that

$$\frac{DF}{AF}=\frac{BF}{FE}.$$

14. A right $\triangle ABC$ with $\angle B$ as right angle. Also $\overline{DA} \perp \overline{AB}$ and $\overline{DE} \perp \overline{AC}$ such that A—E—C. Prove that 1—1 correspondence EDA$\leftrightarrow$BAC is a similarity and $\triangle EDA \sim \triangle BAC$.

15. A quadrilateral ABCD with $\overline{CD} \parallel \overline{AB}$. $\overline{AC}$, $\overline{BD}$ intersect in E. Also given that 1—1 correspondence AED$\leftrightarrow$BEC is a similarity and $\triangle AED \sim \triangle BEC$. Prove that $\overline{AD} \cong \overline{BC}$.

(**Hint.** $\triangle DEC \sim \triangle BEA$).

6. If the sides of two triangles are respectively parallel, then a 1—1 correspondence between the vertices of two triangles could be set up such that it is a similarity.

17. ABCD is a $\parallel^{gm}$. A line through D intersects $\overline{AB}$ in E such that A—E—B and $\overline{CB}$ in F such that C—B—F. Show that AD : AE = FB : BE = FC : CD.

18. ABCD is a $\parallel^{gm}$. M is mid point of $\overline{CD}$. $\overleftrightarrow{BM}$ meets $\overline{AC}$ in L and $\overleftrightarrow{AD}$ in E such that B—L—M—E. Prove that EL = 2BL.

19. Prove that the line joining the mid points of the sides of a triangle is parallel to the third side and half of it in measure.

D

1. Prove that the line joining the mid points of two sides of a triangle forms a triangle which is similar to the given triangle.

2. Prove that the lengths of angle bisectors (segment determined by the vertex and opposite side) of two corresponding angles of two similar triangles are in the same ratio as pair of corresponding sides.

3. Given a $\triangle ABC$. E is a point in $\overline{AB}$ such that AE = EB and D is a point in $\overline{BC}$ such that BD = DC. $\overline{AD}$ and $\overline{CE}$ have a point F in common. Prove that 1—1 correspondence AFC$\leftrightarrow$DFE is a similarity and $\triangle AFC \sim \triangle DFE$.

4. Given a $\triangle ABC$. A point F is in $\overline{AB}$ such that A—F—B and points E, D in $\overline{AC}$ such that A—E—D—C and $\overline{DF} \parallel \overline{BC}$ and $\overline{EF} \parallel \overline{DB}$. Prove that $\frac{AE}{AD}=\frac{AD}{AC}$.

E

1. Write the definition for a pentagon ABCDE to be similar to the pentagon A′B′C′D′E′. Given that pentagon ABCDE$\sim$ pentagon A′B′C′D′E′. Prove that $\triangle ABC \sim \triangle A'B'C'$.

2. Given a $\triangle ABC$ and the altitude $\overline{AD}$ to $\overline{BC}$ such that $B-D-C$ and $\overline{BD} \cong \overline{DC}$. Prove that $1-1$ correspondence $BDA \leftrightarrow CDA$ is a similarity and $\triangle BDA \sim \triangle CDA$.

3. Given a $\triangle ABC$. Its two altitudes $\overline{AD}$ and $\overline{BE}$ intersect in F such that $A-F-D$ and $B-F-E$. Prove that

$$(i)\ \frac{AF}{BF}=\frac{EF}{FD} \quad (ii)\ \frac{AD}{BE}=\frac{AC}{BC}.$$

4. The lengths of corresponding medians of two similar triangles are in the same ratio as the lengths of angle bisectors of angles from whose vertices the medians are drawn.

5. Given a $\triangle ABC$ and a point D in $\overline{AB}$ such that $A-D-B$ and $\overline{AD} \cong \overline{DB} \cong \overline{CD}$. Prove that $\angle C$ is a right angle.

6. Given a $\triangle ABC$. $\angle A$ is a right angle. D is a point in $\overline{AC}$ such that $A-D-C$. E is a point in $\overline{AB}$ such that $A-E-B$. $\overline{DF} \perp \overline{BC}$ also $\overline{EG} \perp \overline{BC}$ such that $B-G-F-C$. Prove that $1-1$ correspondence $CFD \leftrightarrow EGB$ is a similarity and $\triangle CFD \sim \triangle EGB$.

F

1. If C is any point on $\overleftrightarrow{AB}$ such that $A-C-B$ or $A-B-C$ and P is any point on the perpendicular at C to $\overleftrightarrow{AB}$. Prove that the difference of squares of lengths $\overline{AP}$, $\overline{BP}$ is equal to the difference of squares of lengths $\overline{AC}$ and $\overline{BC}$.

2. Given a quadrilateral ABCD such that its diagonals are at right angles to each other. Prove that the sum of the squares of lengths of its two opposite sides equals the sum of the squares of lengths of its other two opposite sides.

3. Given a $\triangle ABC$. $\angle C$ is a right angle. $\overline{CD} \perp \overline{AB}$ such that $A-D-B$ (why ?). E is a point on $\overline{CB}$ such that $C-E-B$ and $\overline{CE} \cong \overline{BD}$. F is a point in $\overline{CA}$ such that $C-F-A$ and $\overline{CF} \cong \overline{AD}$. Prove that $\overline{AE} \cong \overline{BF}$.

4. In any triangle, prove that the sum of the squares on any two sides equals twice the square on half the third side with twice the square on median which bisects the third side.

5. Given a $\triangle ABC$. $\overline{AP} \perp \overline{BC}$; $P \in \overline{BC}$. Prove that $(AB)^2 + BC.CP = (AC)^2 + BC.BP$.

6. ABC is a right triangle with $\angle A$ as a right angle. D, E, F are the mid points of sides $\overline{BC}$, $\overline{AC}$, $\overline{AB}$ respectively. Prove that $2(AD^2+BE^2+CF^2)=3\ BC^2$.

7. In a $\triangle ABC$; $\overline{AD} \perp \overline{BC}$ such that $D \in \overline{BC}$. Also $AD^2 = BD.DC$. Prove that $\angle A$ is a right angle.

8. Given a $\triangle ABC$. $m\angle C > 90$. $\overline{AD} \perp \overline{BC}$ such that $B-C-D$. Prove that $AB^2 = BC^2 + AC^2 + 2\ BC.CD$.

9. Given a $\triangle ABC$. $\overline{AD} \perp \overline{BC}$ such that $B-D-C$. Prove that $AB^2 = BC^2 + AC^2 - 2\ BC.CD$.

10. Given a $\triangle ABC$. $\overline{AD} \perp \overline{BC}$ such that $B-D-C$. E is the middle point of $\overline{BC}$. Prove that $AB^2 - AC^2 = 2\ BC.CD$.

11. Prove that the sum of squares on the sides of a $\|^{gm}$ is equal to the sum of the squares on its diagonals.

(**Hint** : Apply the result of problem IV of this set).

12. Prove that three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on its medians.

13. E is any point in the interior of a rectangle ABCD. Prove that $AE^2 + EC^2 = EB^2 + ED^2$.

14. ABCD is a quadrilateral and E, F are the mid points of its diagonals $\overline{AC}$ and $\overline{BD}$ respectively. Prove that $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4EF^2$.

15. Given a square ABCD. E is a point on $\overline{BC}$ such that $B-C-E$ Prove that $AE^2 = 2\ BC.BE + CE^2$.

16. O is a point in the interior of $\triangle ABC$. $\overline{OD} \perp \overline{BC}$, $OE \perp \overline{CA}$, $\overline{OF} \perp \overline{AB}$. $D \in \overline{BC}$, $E \in \overline{CA}$ and $F \in \overline{AB}$. Prove that $BD^2 + CE^2 + AF^2 = CD^2 + AE^2 + BF^2$.

17. Given a right triangle ABC with $m\angle C = 90$. D is mid point of $\overline{BC}$ and E is mid point of CA. Prove that $4\ (AD^2 + BE^2) = 5\ AB^2$.

18. Show that three times the square on the measure of any side of a triangle equals four times the square on measure of its altitude.

19. ABC is a right triangle with $m\angle C = 90$. p is the length of perpendicular from C to $\overline{BA}$. Show that $pc = ab$ and also

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

20. Given a right isosceles $\triangle$. A point D is on hypotenuse $\overline{BC}$. Prove that $2\ AD^2 = BD^2 + DC^2$.

21. If two medians of a triangle are congruent then prove that triangle is isosceles.

CHAPTER XIII

AREAS OF POLYGONAL REGIONS

After having studied the pythagoras theorem, it is logical to go on to study the concept of area of a triangular region or any polygonal region. The Egyptians were the first to talk about the concept of area. It is said that king Sesostris had divided the available land area amongst all Egyptians in such a way that each one got a piece of land of same size. Every beneficiary of the land had to pay yearly revenues to the king. As most of the Egyptians lived in the vicinity of river Niles, the river when flooded or on changing its course, would tore away some parts of the pieces of land allotted to some of them. They had to inform the king about this who would send observers to measure out by how much the land of an individual beneficiary had become smaller so that he might be required to pay revenues in proportion to land area left with him.

The papyrus written by the Egyptian mathematician Ahmes, which is estimated to have been written in 3400 B.C. and which is considered to be the most ancient mathematical handbook known to the civilization makes a mention of some statements for calculating areas. According to these statements, the area of an isosceles triangle would be calculated by multiplying half the base by one side. The area of an isosceles trapezoid would be calculated by multiplying half the sum of the parallel sides by one of the non-parallel sides. The area of a circle would be $\left(\frac{8}{9} \times \text{diameter}\right)^2$ or $\frac{256}{81}$ $(\text{radius})^2$ or $3{\cdot}16$ radius^2. It means that in 3400 B.C., the Egyptians took π as $3{\cdot}16$ which is really marvellous. It also indicates that 'rope-stretchers' of Egypt would construct a right triangle upon a given line by stretching around three pegs a rope consisting of three parts of lengths in the ratio 3 : 4 : 5.

Ancient Egyptians approach to the subject of geometry or 'areas' can neither be said to be axiomatic nor to be rigorous as they simply formulated some results without any attempt to give logical or rigorous proofs but keeping in view the fact that they knew about these results 5000 years ago, one simply has to praise their ingenuity and their down to earth approach in solving their problems.

We, in this course, however intend to follow axiomatic approach in developing this concept of area and for that purpose, we want to define a triangular region or a polygonal region. Also we propose to state four more postulates so as to achieve our objective.

We have already defined a triangle as a union of three segments determined by three non-collinear points. Every triangle has a set of points which we call the interior of the triangle. A triangular region is the union of points consisting the interior of the triangle and the points that belong to the triangle. The triangle itself is known as the boundary of the triangular region. We may like to extend this to state that a polygonal region is the union of the polygon (a quadrilateral, a rectangle or a square is also a polygon) and the interior of the polygon. The boundary of a polygonal region could be any general polygon—convex or otherwise or may not even be a polygon. For example any of the following figures will be a polygonal region :

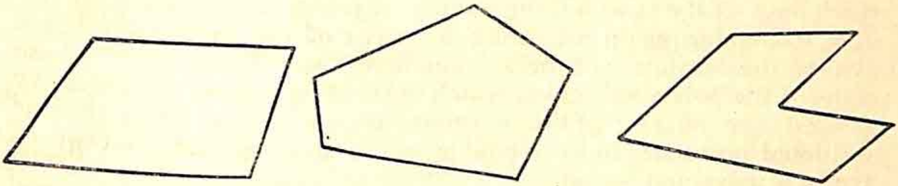


Fig. 209.

Of course we can divide a polygonal region into a set of triangles, but this set of triangular regions forming a particular polygonal region is not unique as would be evident from the following figures :

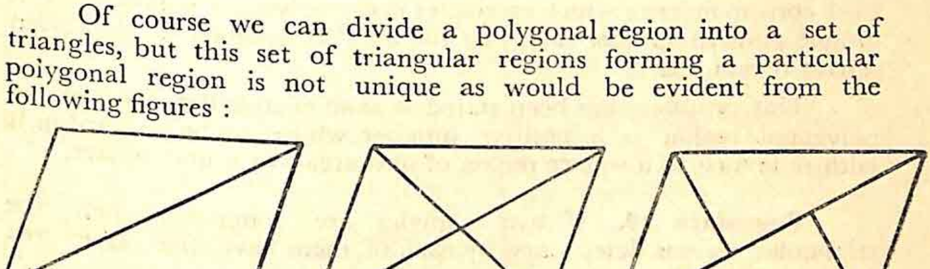


Fig. 210.

Also the boundary of a polygonal region may not be a polygon as will be evident from the following figures :

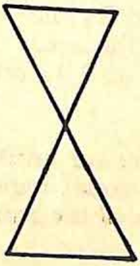


Fig. 211

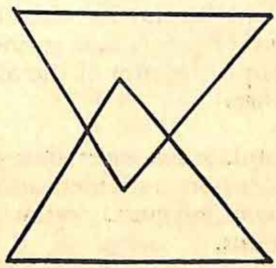


Fig. 212.

We can find a way out to define a polygonal region as a union of a finite number of coplanar triangular regions but we have to make a provision in our definition so that the cases like one represented by those shown in Fig. 212 are eliminated.

Such a situation need to be avoided as when we would postulate about the addition of areas of polygonal region, then such a situation would create trouble for us. Hence we define the polygonal region in the following manner :

Definition XIII (1). A polygonal region is the union of a finite number of coplanar triangular regions, no two of which have any interior points in common.

The union of all those segments that lie in the boundary of one of the constituent triangular regions of the polygonal region and which have at the most a finite number of points in common with any other triangular region which is constituent of the polygonal region is called the *boundary* of the polygonal region. The set of all those points of the polygonal region which doesnot belong to its boundary is called the *interior* of the polygonal region. Now we state four additional postulates so as to enable us to develop the concept of area of a polygonal region.

Postulate 18. If S is any square region, then there exists a 1—1 correspondence which associates every polygonal region with a unique positive number such that the number assigned to the given square region S is 1.

This postulate has been stated so as to express the area of any polygonal region as a positive number which can be assigned to it with reference to a square region of unit area—or a unit square.

Postulate 19. If two triangles are congruent then the triangular regions determined by each of them have the same area relative to a unit square.

This postulate has been stated with a view to put in black and white our intutive conception that two regions of same shape should have the same area wherever they may be situated.

Postulate 20. If a polygonal region R is the union of two polygonal regions R_1 and R_2 such that R_1 and R_2 intersect in atmost a finite number of points and segments then the area of R relative to a unit square is the sum of the areas of R_1 and R_2 relative to the same unit square.

This postulate has been stated to express our intutive idea that if a polygonal region is divided into two polygonal regions then the area of the given polygonal region is the sum of the areas of the two polygonal regions.

Postulate 21. If S is any square region whose sides are of length 1, then the area of any rectangular region R, relative to the unit square S is the product of its two consecutive sides.

Our scheme has been to postulate for (*a*) A unit square (*b*) Area of a polygonal region is a positive number (*c*) Two polygonal regions determined by two congruent triangles have the same area (*d*) Area of any polygonal region can be computed by adding areas of polygonal regions or triangular regions into which it can be divided and (*e*) Area of a rectangular region is the product of its consecutive sides.

We will use these postulates in devising formulae for computing the areas of a right triangle, any triangle, a parallelogram, a quadrilateral etc. etc. However we should note that we compute the area of a region bounded by a triangle or a quadrilateral or a polygon and not the area of a triangle, quadrilateral or a polygon.

Theorem XIII (1). The area of the region bounded by a right triangle is half the product of the measures of its legs.

Given : A right triangle ABC with $\angle B$ a right angle.

To prove that : The area if the region bounded by $\triangle ABC$ equals $\frac{1}{2}\,ac$ or $A=\frac{1}{2}\,ac$, if A is the required area. (In our usual notation $BC=a$ and $AB=c$ etc.)

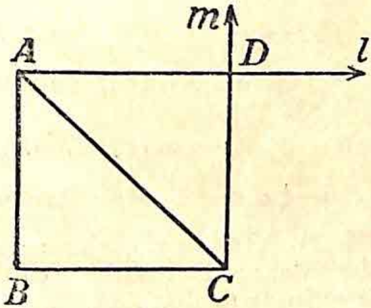


Fig. 213.

Proof.

Statement	*Reason*
(1) Through A, there is a line $l \parallel \overline{BC}$	(1) Parallel Postulate
(2) Through C, there is a line $m \parallel \overline{AB}$	(2) Parallel postulate
(3) $l \nparallel m$	(3) If $l \parallel m$ then $m \parallel \overline{BC}$ as $l \parallel \overline{BC}$. But $m \parallel \overline{AB}$. Hence m would be parallel to two lines $\overline{AB}$, $\overline{BC}$ intersecting at B which will contradict parallel postulate.

(4) l and m have a point D in common	(4) Statement 3
(5) ABCD is a $\parallel^{gm}$	(5) Statements 1 and 2
(6) $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$	(6) Opposite sides of a $\parallel^{gm}$
(7) $\angle B \cong \angle D$	(7) Opposite angles of a $\parallel^{gm}$
(8) $1-1$ correspondence $ABC \leftrightarrow ADC$ is a congruence and $\triangle ABC \cong \triangle ADC$	(8) S.A.S. Postulate
(9) Area of each of $\triangle ABC$, $\triangle ADC$ is equal to A	(9) Postulate 19
(10) Area of $\parallel^{gm}$ ABCD is 2A	(10) Postulate 20
(11) $\angle B$ is a right angle	(11) Given
(12) $\angle D$ is a right angle	(12) Statement 7
(13) $\angle A \cong \angle C$	(13) Opposite angles of a $\parallel^{gm}$
(14) Each of $\angle A$ and $\angle C$ is a right angle	(14) Why ?
(15) ABCD is a rectangle	(15) Why ?
(16) Area of a rectangle $ABCD = BC.AB$	(16) Postulate 21
(17) $2A = BC.AB$	(17) Statements 10 and 16
(18) $A = \frac{1}{2} BC.\ AB = \frac{1}{2}\, a.c$	(18) Statement 17.

Theorem XIII (2). The area of the region bounded by any triangle is equal to half the product of the measures of any one side of a triangle and the altitude to that side.

Given : A $\triangle ABC$. h is the length of $\overline{AD}$, the altitude from A to $\overline{BC}$, where $D \in \overline{BC}$.

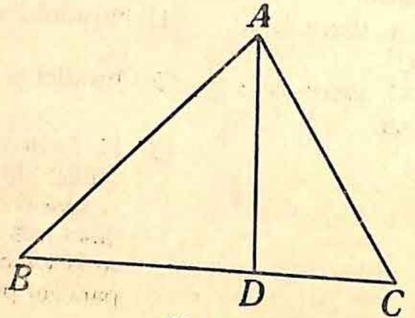


Fig. 214.

To prove that ; Area of the region determined by $\triangle ABC$ $= \frac{1}{2}\, h.\, \overline{BC}$.

Proof. There are three alternatives about the position of D on $\overleftrightarrow{BC}$. (*i*) B—D—C (*ii*) D=B or D=C (*iii*) D—B—C.

First Case. B—D—C.

Statement	*Reason*
(1) Area of the region bounded by right triangle ABD $= \frac{1}{2}\, h.BD$	(1) Theorem XIII (1)
(2) Area of the region bounded by $\triangle ACD$ $= \frac{1}{2}\, h.CD$	(2) Theorem XIII (1)
(3) Area of the region bounded by $\triangle ABC$ is $\frac{1}{2}\, h.BD + \frac{1}{2}\, h.CD$ $= \frac{1}{2}\, h(BD+CD)$ $= \frac{1}{2}\, h.BC$.	(3) Postulate 20, Distributive law, B—D—C

Second Case. When D=B or D=C. This case is covered by Theorem XIII (1)

Third Case. When D—B—C (or B—C—D).

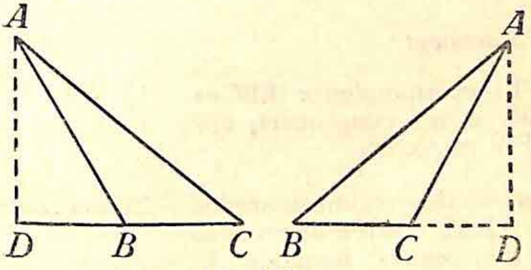


Fig. 215.

Statement	*Reason*
(1) Area of region bounded by right $\triangle ABD = \frac{1}{2}h.BD$ and that bounded by right $\triangle ADC = \frac{1}{2}h.DC$	(1) Theorem XIII (1)
(2) Area of the region bounded by $\triangle ACD$ equals the sum of the areas of regions bounded by $\triangle ABC$ and $\triangle ADB$	(2) Postulate 20

(3) $\frac{1}{2}h.DC=$ Area of $\triangle ABC+\frac{1}{2}h.BD$	(3) Statement 2
(4) Area of $\triangle ABC=$ $\frac{1}{2}h.DC-\frac{1}{2}h.BD.$ $=\frac{1}{2}h(DC-BD)$ $=\frac{1}{2}h.BC.$	(4) Statement 3 and D−B−C.

It may however be noted that we can prove this theorem by taking into consideration any one of the sides of triangle and the altitude from the opposite vertex to that side.

Theorem XIII (3). The area of the region bounded by any $\parallel^{gm}$ is the product of measures of any one of its sides and the altitude from the opposite vertex to that side.

Given : A $\parallel^{gm}$ BCDE . $\overline{EF}$ is the altitude from E to $\overline{BC}$. ($\overline{DG}$ will be the altitude from D to $\overleftrightarrow{BC}$ but as $\overline{EF}\cong\overline{DG}$, we can consider either one of them) Also $EF=h$.

To prove that : Area A of $\parallel^{gm}$ BCDE$=h.BC$.

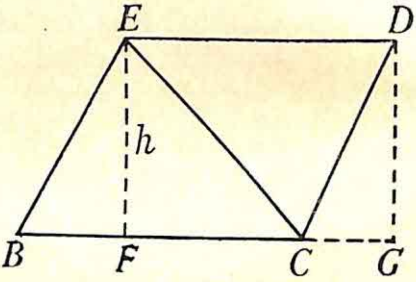


Fig. 216.

Proof.

Statement	*Reason*
(1) 1−1 correspondence EBC↔CDE is a congruence and $\triangle EBC\cong\triangle CDE$.	(1) Why ?
(2) Area of the region bounded by $\triangle EBC$ is same as the area of the region bounded by $\triangle CDE$	(2) Postulate 19
(3) Area of the region bounded by $\triangle EBC$ is $\frac{1}{2}h.BC$	(3) Theorem XIII (2)
(4) Area of the region bounded by $\parallel^{gm}$ BCDE is the sum of the areas of the regions bounded by $\triangle EBC$ and $\triangle CDE$	(4) Postulate 20
(5) $A=\frac{1}{2}h.BC+\frac{1}{2}h.BC=\frac{1}{2}h.2BC$ $=h.BC$	(5) Statement 4.

Definition XIII (2). (*Trapezoid or Trapezium*). It is a quadrilateral in which two and only two sides are parallel.

Theorem XIII (4). The area of the region bounded by a trapezoid is half the product of its altitude and the sum of its parallel sides.

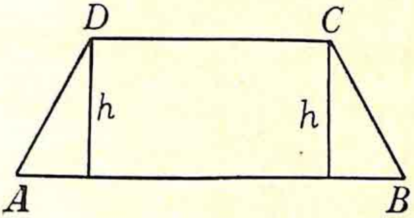


Fig. 217.

Given . A trapezoid ABCD in which $\overline{AB} \parallel \overline{CD}$ and h is the length of its altitude from D or C to $\overline{AB}$ (why is it same ?)

To prove that : Area of trapezoid ABCD is $\frac{1}{2}\, h(AB+CD)$ or is $\frac{1}{2}h((b_1+b_2)$ if $AB=b_1$ and $CD=b_2$.

Proof.

Statement	*Reason*
(1) Area of the region bounded by $\triangle ABD=\frac{1}{2}\, h.b_1$	(1) Theorem XIII (2)
(2) Area of the region bounded by $\triangle BCD=\frac{1}{2}\, h.b_2$	(2) Theorem XIII (2) How?
(3) Area of the trapezoid ABCD equals the sum of the areas of the regions bounded by $\triangle ABD$ or $\triangle BCD$	(3) Postulate 20
(4) Area of trapezoid ABCD= $\frac{1}{2}\, h\, b_1+\frac{1}{2}\, hb_2=\frac{1}{2}\, h(b_1+b_2)$	(4) Statement 3.

Theorem XIII (5). The areas of triangular regions bounded by similar triangles are proportional to the squares of measures of any pair of corresponding sides.

Given : $\triangle BCD$ and $\triangle EFG$ such that $1-1$ correspondence $BCD \leftrightarrow EFG$ is a similarity and $\triangle BCD \sim \triangle EFG$.

To prove that : $\frac{A}{A'}=\frac{d^2}{g^2}=\frac{c^2}{f^2}=\frac{b^2}{e^2}$ where A and A′ denote areas of $\triangle BCD$ and $\triangle EFG$ and b, c, d, e, f, g have the usual meaning.

Proof. Let h be the measure of altitude from B to $\overline{CD}$ and h' be the measure of altitude from E to $\overline{FG}$.

Statement	*Reason*
(1) $d=kg$; $c=kf$; $b=ke$ and $h=kh'$	(1) If k is the constant of proportionality. Why ?

(2) $A=\frac{1}{2}\,h.b$ and $A'=\frac{1}{2}\,h'.e$	(2) Why ?
(3) $A=\frac{1}{2}h.b=\frac{1}{2}kh'.ke=\frac{1}{2}k^2h'e$ $=k^2A'$	(3) Statements 1 and 2
(4) $\frac{A}{A'}=k^2$	(4) Statement 3
(5) $k=\frac{d}{g}$, hence $k^2=\frac{d^2}{g^2}=\frac{c^2}{f^2}=\frac{b^2}{e^2}$	(5) Statement 1
(6) $\frac{A}{A'}=\frac{d^2}{g^2}=\frac{c^2}{f^2}=\frac{b^2}{e^2}$	(6) Statement 5.

Problem Set

1. If one side of a $\triangle ABC$ is congruent to one side of $\triangle DEF$ and if the altitudes to each of these congruent sides are also congruent then the areas of the regions bounded by each of the triangles are the same.

2. Given a $\triangle ABC$. D is a point on $\overline{BC}$ such that $\overline{BD}\cong\overline{DC}$. There is a point E on $\overleftrightarrow{AD}$ such that $A-D-E$. Prove that the area of the region bounded by $\triangle AEB$ is the same as the area of the region bounded by $\triangle AEC$.

3. Prove that the diagonals of a $\|^{gm}$ divide the region bounded by it into four triangular regions which have equal area.

4. Obtain the formula for finding the area of a region bounded by a rhombus (Rhombus is a $\|^{gm}$ whose all sides are congruent in pairs) if the measure of one of its sides and the measure of one of its diagonals is given.

(**Hint.** First prove that the diagonals bisect each other at right angles).

5. A convex quadrilateral ABCD is called a kite when $\overline{AB}\cong\overline{BC}$ and $\overline{CD}\cong\overline{DA}$. Obtain a formula to find the area of the region bounded by the kite if the measures of its diagonals are given.

(**Hint** : First prove that the diagonals are perpendicular to each other).

6. Any median of a triangle divides it into two triangles whose areas are the same.

CHAPTER XIV

ANOTHER MODERN GEOMETRY

(*Analytical Plane Geometry*)

In the previous chapters, we have discussed the modern synthetic Geometry as an Axiomatic System. In this chapter we intend to introduce another modern geometry—analytical plane geometry or co-ordinate geometry.

Analytical or co-ordinate geometry was developed by Rene' Descartes (1596—1650) who was the first modern mathematician and philosopher to discard traditions and errors of the earlier times. Descartes felt that the geometeres of the early times used obscure and difficult methods in solving geometry problems. Descartes synthesised geometry and Algebra and the result was the system of analytical geometry.

In the chapter on 'Distance and Measurement' we learnt how to set up 1—1 correspondence between the set of real numbers and the points on a line. In other words we learnt to set up a co-ordinate system on a line. In this we assigned a unique real number to every point on the line. The real number that is assigned to a point is called its co-ordinate. In this way we assign a co-ordinate to every point of a line and a point on the line is determined when we name its co-ordinate provided we know the starting point or the point that is assigned the co-ordinate zero. This point of reference or the point whose co-ordinate is zero is called *origin*.

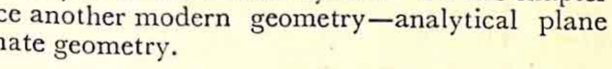


Fig. 218.

In the plane co-ordinate geometry, we assign co-ordinates to any point in a plane. In the case of a point on a line, a single real number can serve as a co-ordinate and if we know it, we can locate the required point on the line. Of course, we have to know the origin and the unit of measurement. In the case of a point in a plane, a single real number will not help us. We will have to know two things (1) The line on which the given point lies (2) The co-ordinate of the point on the given line. To solve this difficulty two lines in the plane are taken as lines of reference which are generally

called the axes of reference. Generally
a way that these are perpendicular to
need not to be perpendicular). The
these lines is taken to be the origin.

Let X and Y be the lines which

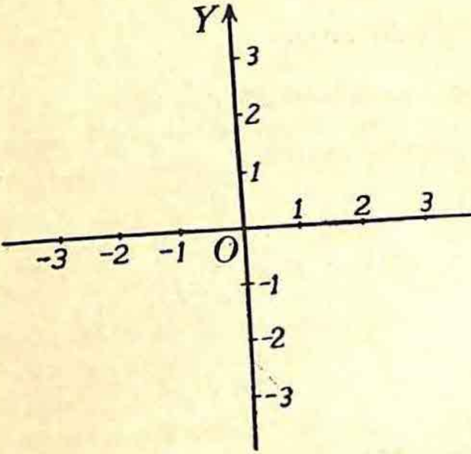


Fig. 219.

Suppose we have a point A in
know that we have a unique perpen
X (Why ?) and a unique perpenc
cular $\overline{AC}$ from A to the line
If AC$=a$ and AB$=b$ then $(a,$
are called the co-ordinates of t
point A. Measure of perpendicul
segment from the given point to t
Y-axis is called the X-co-ordinate
the point or abscissa of the poi
while the measure of perpendicul
segment from the point to the X-a
is called the Y-co-ordinate of
point or the ordinate of the point.
the case of the point A, a is the X-
ordinate or abscissa of the point
while b is the Y-co-ordinate or
ordinate of the point A. While
point, we always write (by conver
first and Y-co-ordinate of the poin
of a point are indicated by an 'ord
the X-co-ordinate coming first
next.

If we are given that the co-c

these lines are selected in such
each other. (Of course these
point that is common between

are perpendicular to each other and let O be the origin. Let a co-ordinate system be set up on each of these lines with O as the origin or the point of reference. The line X is called the X-axis and the line Y is called the Y-axis. Now we are in a position to locate any point in the plane with the help of a pair of numbers —a number indicating its perpendicular distance from Y-axis and another number indicating its perpendicular distance from X-axis.

the plane of this paper. We
dicular $\overline{AB}$ from A to the line
li-
Y.
b)
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dicating the co-ordinates of a
on) X-co-ordinate of the point
next. In fact the co-ordinates
red pair' of real numbers with
and the Y-co-ordinate coming

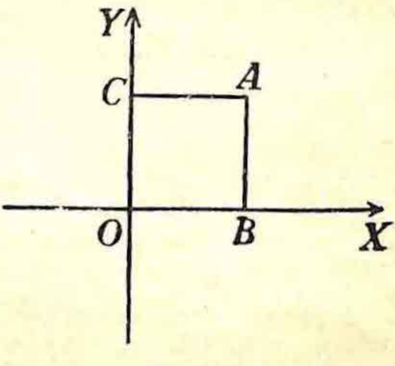


Fig. 220.

inates of a point are (3, 4) then

it implies that after starting from O the origin, go 3 units along the X-axis in the direction which is taken to be the positive direction of the X-axis and then move 4 units along a line perpendicular to X-axis at that point and in the direction which is taken to be the positive direction of Y-axis. If $\overleftrightarrow{WE}$ divides the plane of the paper into two half planes H_1 and H_2 such that $N \in H_1$ and $S \in H_2$ (as in this figure) and if $\overleftrightarrow{NS}$ divides the plane of the paper into two half planes H_3 and H_4 such that $E \in H_3$ and $W \in H_4$ then it is a convention that Y-co-ordinate of a point belonging to H_1 is taken to be positive while the Y-co-ordinate of a point belonging to H_2 is taken to be negative. Also the X-co-ordinate of a point belonging to H_3 is taken to be positive while the X-co-ordinate of a point belonging to H_4 is taken to be negative. So if a point belongs to $H_1 \cap H_3$ then its both the co-ordinates will be positive but if it belongs to $H_2 \cap H_4$ its both the co-ordinates will be negative. If a point belongs to $H_2 \cap H_4$, its X-co-ordinate will be positive while its Y-co-ordinate will be negative but if a point belongs to $H_1 \cap H_3$ then its X-co-ordinate will be negative while its Y-co-ordinate will be positive. As an illustration the co-ordinates of A, B, C, D in this figure are

A(2, 4) ; B(−2, 3) ; C(−3, −4) ; D(3, −5)

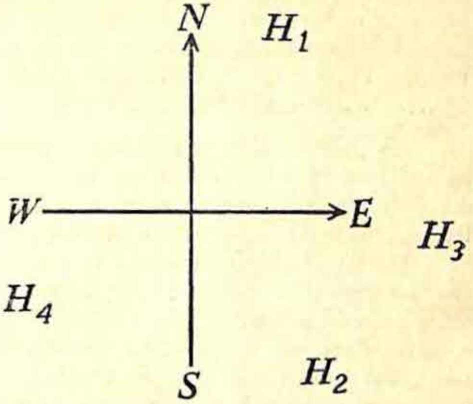


Fig. 221.

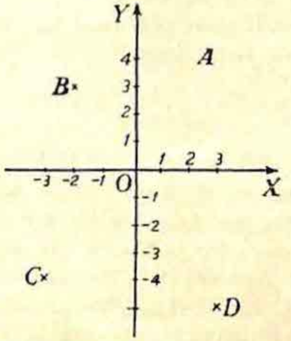


Fig. 222.

A plane is a two dimensional space while a line is a one dimensional space. In fact a line is a one-dimensional space continuum (in the words of one of the very few top-most mathematicians, Prof. Albert Einsten—the author of 'Theory of Relativity'). A distinguishing characteristics of a continuum is that the interval separating any two points may be divided into an infinite number of arbitrarily small steps. Theoretically we can locate an infinite number of points between any two points on a line howsoever close those may be to each other and we can assign co-ordinates to these points. If we have any two real numbers a and b such that $a>b$ but $a+1<b$ then we have an infinite number of real numbers each of which will be less than a but greater than b.

A railway line is a physical model for a one-dimensional space as a driver of a train can describe his position at any time by citing a single co-ordinate—a station or a milestone, (provided of course he travels only on one line) but a sailor sailing in the sea cannot do so. He must think in terms of two dimensions as the surface of the sea is a two dimensional continuum for the purposes of locating a ship somewhere sailing in the sea. The co-ordinate points by which a sailor fixes his position in his two dimensional continuum are latitude and longitude. A pilot flying his plane will need three co-ordinates to help him locate the position of his plane with reference to the airport from which he started his flight as the space very near the earth can be termed as a three-dimensional continuum. The pilot must indicate that at a given time, the plane is at a latitude X, longitude Y and altitude Z. It is essential that traffic manager sitting in the airport must know the time at which the plane is at latitude X, longitude Y and altitude Z. So for the traffic manager, the flight of a plane is in four dimensions. Of course the time co-ordinate is the fourth co-ordinate. The traffic manager, would visualise the flight of a plane in four-dimensional space time continuum. Our world is a space-time continuum. This idea of fourth dimension time-dimension was given by Prof. Einstein. Now the mathematicians talk in terms of n-dimensional space $(n>4)$. However we will restrict ourselves to $n=2$ or only to a two-dimensional space. Earlier we have stated that any point in a plane can be represented by an ordered pair of real numbers. The converse also holds. However we state it in the form of a following postulate.

Postulate 22. To every point in the plane, there corresponds a unique ordered pair of real numbers and to every ordered pair of real number there corresponds a unique point in the plane.

The co-ordinate system thus set up is known as a system of cartesian co-ordinates (the term cartesian is derived from Descartes) or Rectangular co-ordinates (as the axes are at right angles). It is to be noted that on the X-axis, points to the right of origin are represented by positive numbers and those to the left of it are represented by negative numbers ; on the Y-axis the points in the upward direction from origin are represented by positive numbers and those in the downward direction from the origin are represented by negative numbers.

In Analytic Geometry, unlike the Synthetic Geometry the terms like point, line etc. are not left undefined. A Point is an algebraic entity-defined as an ordered pair of numbers. Certain sets of points will be called lines. Certain other sets of points will be called planes etc.

Plotting points on a Graph Paper. For plotting points or for drawing figures in Co-ordinate Geometry a printed graph paper is ordinarily taken. Alternatively a rubber stamp of a graph paper can be prepared and the same could be stamped on a plane paper. In the middle of it, X-axis and Y-axis are drawn and labelled.

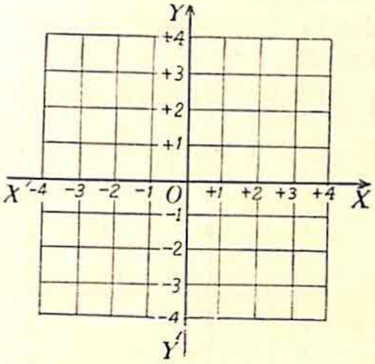


Fig. 223.

The origin is denoted by O. Then the points with co-ordinates (1, 0) and (0, 1) are labelled in order to indicate the unit to be used. The co-ordinates of the origin are (0, 0). It may however be noted that X-co-ordinate of all the points lying on any line parallel to Y-axis is the same and Y-co-ordinate of any point lying on a line parallel to X-axis is the same.

The distance between any two points :

It is apparent from the figure that the distance between A and B which have the same Y-co-ordinate (ordinate) will be the difference of their abscissae. The distance between A and B=5—1=4. Also the distance between B and C which have the same Xc-o-ordinate (abscissa) will be given by the difference between their ordinates.

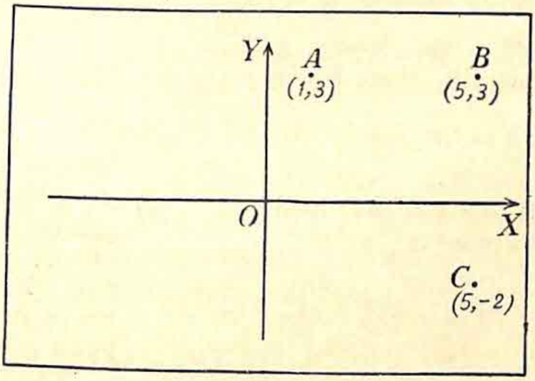


Fig. 224.

The distance between B and $C = 3-(-2) = 5$. How to find the distance between two points with unequal co-ordinates. For this we prove the following theorem.

Theorem XIV (1). The distance between two points $P_1(x_1, y_1)$ and $P_2(x_2 y_2)$ will be given by the formula

$$d(P_1P_2) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Given : A point P_1 with co-ordinates (x_1, y_1) and a point P_2 with co-ordinates (x_2, y_2).

To prove that : $d(P_1P_2)$ [distance between P_1 and P_2] equals

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

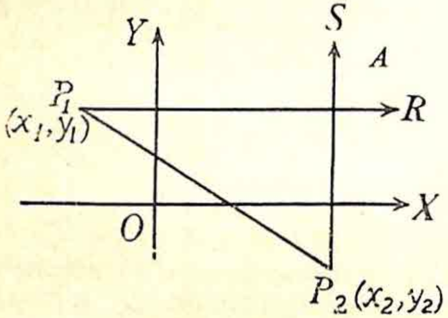


Fig. 225.

Proof.

Statement	*Reason*
(1) Through P_1, there is a ray $\overrightarrow{P_1R}$ ∥ to the X-axis and through P_2, there is a ray $\overrightarrow{P_2S}$ ∥ to the Y-axis	(1) Parallel Postulate
(2) $\overrightarrow{P_1R}$ and $\overrightarrow{P_2S}$ have a common point A	(2) If $P_1R \parallel P_2S$ then X-axis should be ∥ to Y-axis which is not true. Therefore $P_1R \nparallel P_2S$ and hence have a common point
(3) $\angle P_1AP_2$ is a right angle	(3) Why ? It is because X-axis and Y-axis are perpendicular to each other

(4) Co-ordinates of A are (x_2, y_1)	(4) Why ?
(5) $m(P_1A) = \mid x_2 - x_1 \mid$	(5) When the ordinates of two points are same then the distance between them is given by the difference between their abscissae
(6) $m(\overline{P_2A}) = \mid y_2 - y_1 \mid$	(6) Why ?
(7) $[d(\overline{P_1P_2})]^2 = (\mid x_2 - x_1 \mid)^2 + (\mid y_2 - y_1) \mid)^2$	(7) Pythagorean relation
(8) $(\mid x_2 - x_1 \mid)^2 = (x_2 - x_1)^2$ $(\mid y_2 - y_1 \mid)^2 = (y_2 - y_1)^2$	(8) Why ?
(9) $[d(\overline{P_1P_2})] = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.	(9) Statements 7 and 8.

It may be noted that distances between the points are taken to be non-negative.

Example 1. Find the distance between the points P_1 and P_2 with co-ordinates $(3, -7)$ and $(-10, 3)$ respectively.

Solution. $d(P_1P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Now $(x_2 - x_1) = (-10) - (3) = -13$

$(y_2 - y_1) = 3 - (-7) = 10$

$\therefore$ $d(P_1P_2) = \sqrt{(-13)^2 + (10)^2} = \sqrt{169 + 100} = \sqrt{269}$.

Example 2. Find the distance between the points P_1 and P_2 with co-ordinates $(-3, -5)$ and $(7, -7)$ respectively.

Solution. $d(P_1P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Now $(x_2 - x_1) = 7 - (-3) = 10$

$(y_2 - y_1) = -7 - (-5) = -2.$

$\therefore$ $d(P_1P_2) = \sqrt{(10)^2 + (-2)^2}$

$= \sqrt{100 + 4} = \sqrt{104} = 2\sqrt{26}$.

Example 3. Find the distance between the origin and a point P whose co-ordinates are $(-7, 3)$.

Solution. Now the co-ordinates of origin are $(0, 0)$

$(x_2 - x_1) = -7 - (0) = -7$

$(y_2 - y_1) = 3 - 0 = 3$

$$d(P_1P_2)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$=\sqrt{(-7)^2+(3)^2}=\sqrt{49+9}=\sqrt{58}.$$

Example 4. Find the distance of a point $P(x_1, y_1)$ from the origin.

The co-ordinates of the origin are (0, 0)

$$\text{distance}=\sqrt{(x_1-0)^2+(y_1-0)^2}$$
$$=\sqrt{x_1^2+y_1^2}$$

Theorem XIV (2). Prove that the co-ordinates of the mid point of the line segment $\overline{P_1P_2}$, the co-ordinates of whose end points are (x_1, y_1) and (x_2, y_2) are

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Given : A line segment $\overline{P_1P_2}$ whose end points are $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

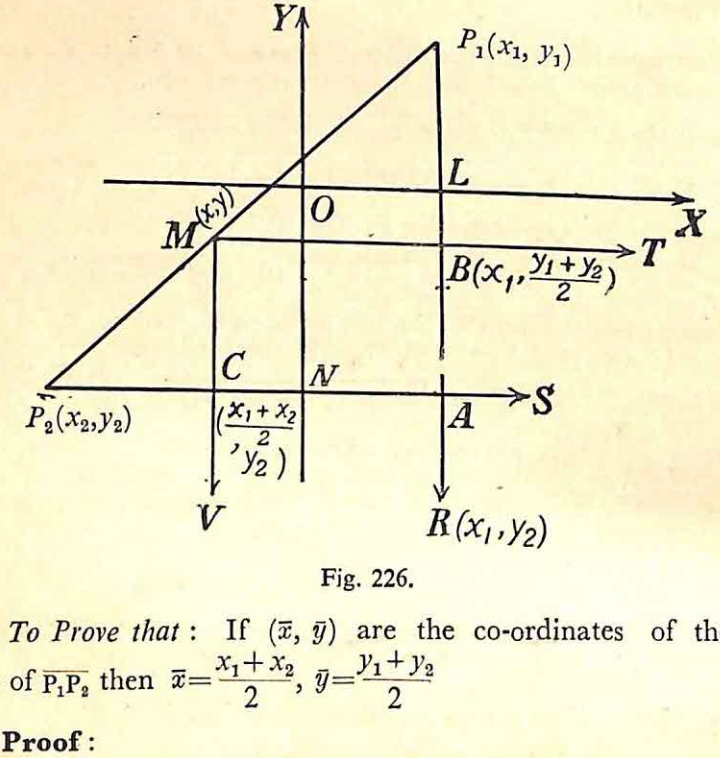


Fig. 226.

To Prove that : If $(\bar{x}, \bar{y})$ are the co-ordinates of the mid point of $\overline{P_1P_2}$ then $\bar{x}=\frac{x_1+x_2}{2}$, $\bar{y}=\frac{y_1+y_2}{2}$

Proof :

Let M be the mid point of $\overline{P_1P_2}$. Let the co-ordinates of M be $(\bar{x}, \bar{y})$.

Statement	*Reason*
(1) There is a ray $\overrightarrow{P_2S} \parallel$ to X-axis and a ray $\overrightarrow{P_1R} \parallel$ to Y-axis	(1) Why ?
(2) $\overrightarrow{P_1R}$ and $\overrightarrow{P_2S}$ have a common point A	(2) Why ?
(3) The co-ordinates of A are (x_1, y_2)	(3) Why ?
(4) Through M, there is a ray $\overrightarrow{MT} \parallel \overrightarrow{P_2S}$	(4) Why ?
(5) M_2T has a point B in common with $\overline{P_1A}$	(5) Pasch's Postulate as $\overleftrightarrow{MT}$ does not pass through A or P_1
(6) $\overrightarrow{MV}$ is a ray through M $\parallel \overrightarrow{P_1A}$ having a point C in common with $\overrightarrow{P_2S}$	(6) Why ?
(7) $P_1A = y_1 + y_2$ and $P_2A = x_1 + x_2$	(7) $P_1A = P_1L + LA$ and $P_2A = P_2N + NA$
(8) $\overline{P_1B} \cong \overline{BA}$ and $\overline{P_2C} \cong \overline{CA}$	(8) $\triangle P_1MB \cong \triangle MP_2C$ Why ?
(9) Co-ordinates of B are $\left(x_1, \frac{y_1+y_2}{2}\right)$ and co-ordinates of C are $\left(\frac{x_1+x_2}{2}, y_2\right)$	(9) Statements 7 and 8
(10) The co-ordinates of M are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$	(10) Statement 9

Example 5. Find the co-ordinates of the mid point of line segment $\overline{P_1P_2}$, the co-ordinates of whose end points are $(-3, 7)$, $(5, -6)$.

Solution.

$$\bar{x} = \frac{x_1 + x_2}{2} = \frac{-3+5}{2} = 1$$

$$\bar{y}=\frac{y_1+y_2}{2}=\frac{7+(-6)}{2}=\frac{1}{2}$$

$\therefore$ The co-ordinates of mid point are $(1, \frac{1}{2})$.

Example 6. The co-ordinates of mid point of a line segment with end points P_1, P_2 are $(7, -3)$. Given that the co-ordinates of P_1 are $(13, 4)$, what shall be the co-ordinates of P_2 ?

Solution.

Let (x_2, y_2) be the co-ordinates of P_2 then we know that

$$\bar{x}=\frac{x_1+x_2}{2} \text{ and } \bar{y}=\frac{y_1+y_2}{2}$$

$$\therefore \quad 7=\frac{13+x_2}{2} \text{ and } -3=\frac{4+y_2}{2}$$

or $\quad 14=13+x_2$ and $-6=4+y_2$

or $\quad x_2=1$ and $y_2=-10$

$\therefore$ co-ordinates of P_2 are $(1, -10)$

Example 7. Find the area of a region bounded by a triangle whose vertices are $(10, 0)$, $(14, 8)$ and $(-4, 0)$.

Solution.

It is clear that $\angle ABC$ is obtuse hence the perpendicular from

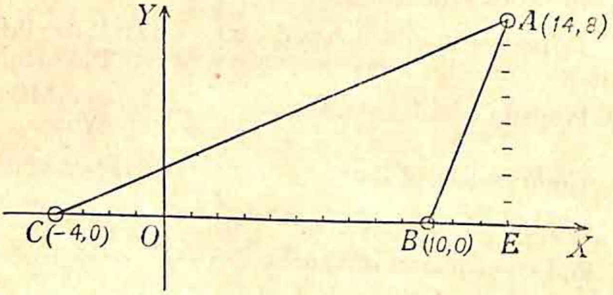


Fig. 227.

A to the base $\overline{CB}$ shall be $\overline{AE}$ such that C–B–E. Also area of region bounded by $\triangle ABC$ equals the area of region bounded by $\triangle AEC$ – Area of region bounded by $\triangle AEB$ (Postulate 21). Now Area of region bounded by $\triangle AEC=\frac{1}{2}$ CE.AE$=\frac{1}{2}.18.8=72$ square-units. Area of region bounded by $\triangle AEB=\frac{1}{2}$ BE.AE$=\frac{1}{2}.4.8=16$ square-units Hence Area of region bounded by $\triangle ABC=72-16=56$ square-units.

Problem Set

1. Plot the points P(0, 0), Q(8,0), R(0, 6), find the length of each side of $\triangle PQR$. Is the triangle right angled ? What is its

area ? If E is the mid point of $\overline{PR}$ and F is the mid point of $\overline{QR}$, find the co-ordinates of E and F. Find the length of $\overline{EF}$. Is it half of the length of $\overline{PQ}$?

2. Find the length of sides of a triangle whose vertices are A(7, 9), B(9, 7), C(0, 0).

3. Show that a triangle whose vertices are P(−8, 0), Q(0, 8), R(8, 0) is isosceles triangle.

4. Find the mid points of line joining (*i*) A(7, 5) and B(3, −9) (*ii*) P(8, 0) and Q(0, 8) (*iii*) A(0, 0) and B(−6, 6).

5. Find the length of median from A to $\overline{BC}$ of a $\triangle$ABC, the co-ordinates of whose vertices are A(−8, 0), B(8, 0) and C(0, 10).

6. Find the length of median from A to $\overline{BC}$ of a $\triangle$ABC, the co-ordinates of whose vertices are A(5, 8), B(−4, 6) and C(−2, −10).

7. Show that the points A(7, 10), B(−5, −6) and C(4, 6) are collinear.

8. Find the perimeter of a rectangle ABCD, three of whose vertices are A(—4, −2), B(−2, 2) and C(10, −4).

9. The distance of a point P(x, y) from the origin equals the distance between origin and the point (3, 2). Prove that $x^2+y^2-13=0$ is a true statement.

10. Show that the $\triangle$PQR, the co-ordinates of whose vertices are P(3, 3), Q(9, 0) and R(12, 21) is a right triangle.

(**Hint** : Use the converse of the Pythagorean theorem)

11. Show that the point B is between A and C if the co-ordinates of A, B, C are A(7, 10), B(4, 6) and C(−5, −6).

(**Hint** : Prove that AB+BC=AC)

12. Find the lengths of diagonals of a quadrilateral whose vertices are A(6, 4), B(−1, 2), C(3, −2) and D(2, 8).

13. Graph the line segment between each of the pair of points whose co-ordinates are given below. Then compute the mid point of the segments and check your answer from the graph.

(*i*) (0, 0) and (−8, 0), (*ii*) (4, 9) and (−4, 11) (*iii*) (−15, 3) and (−6, −1), (*iv*) (1, 1) and (17, 1) (*v*) (−6, 3) and (−11, 12)

14. Show that P(12, 8), Q(−2, 6) and R(6, 0) are the vertices of a right triangle with R as a right angle.

15. If (3, 2) and (−3, 2) be the two vertices of an equilateral triangle and if the origin lies in the interior of the triangle, find the co-ordinates of the third vertex.

[**Hint** : Let (x, y) be the co-ordinates : Then the measure of each of its sides should be the same.]

Dividing a line segment in a given ratio

Theorem XIV (3). The co-ordinates of a point which divides the line segment with end points A(x_1, y_1) and B(x_2, y_2) in the ratio $p : q$ are $\left(\frac{px_2+qx_1}{p+q}, \frac{py_2+qy_1}{p+q}\right)$

Given : A(x_1, y_1) and B(x_2, y_2) and a point C on $\overline{AB}$ such that

$$\frac{m\,\overline{AC}}{m\,\overline{CB}}=\frac{p}{q}$$

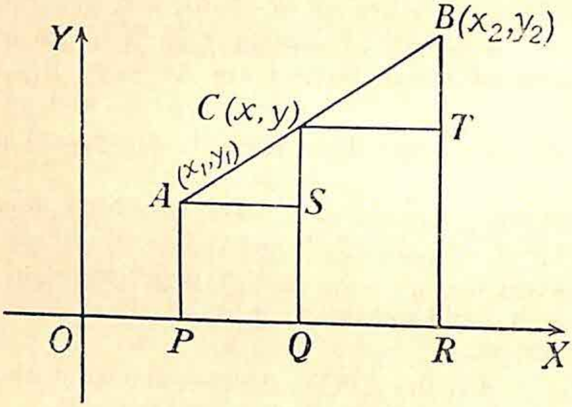


Fig. 228.

To Prove that : The co-ordinates of

$$\text{C are } \left(\frac{px_2+qx_1}{p+q}, \frac{py_2+qy_1}{p+q}\right)$$

Proof : Let the co-ordinates of C be (x, y)

Statement	*Reason*
(1) Through A,B,C there are line segments $\overline{AP}$, $\overline{CQ}$ and $\overline{BR}$ each parallel to Y-axis such that P, Q, R are on X-axis	(1) Parallel Postulate
(2) Through A, there is a line segment $\overline{AS} \parallel$ to X-axis and having a point S in common with $\overline{CQ}$ and through C there is a line segment $\overline{CT} \parallel$ to Y-axis and having a point T in common with $\overline{BR}$	(2) Why ?

(3) 1−1 correspondence CAS↔BCT is a similarity and $\triangle CAS \sim \triangle BCT$	(3) Why ?
(4) $\frac{m\ \overline{AS}}{m\ \overline{CT}}=\frac{m\ \overline{CS}}{m\ \overline{BT}}=\frac{m\ \overline{AC}}{m\ \overline{CB}}$	(4) Statement 3
(5) $m\ \overline{AS}=x-x_1$	(5) $m\ \overline{AS}=m\ \overline{PQ}=m\ \overline{OQ}-m\ \overline{OP}$
(6) $m\ \overline{CT}=x_2-x$	(6) $m\ \overline{CT}=m\ \overline{QR}=m\ \overline{OR}-m\ \overline{OP}$
(7) $m\ \overline{CS}=y-y_1$	(7) $m\ \overline{CS}=m\ \overline{CQ}-m\ \overline{SQ}=m\ \overline{CQ}-m\ \overline{AP}$
(8) $m\ \overline{BT}=y_2-y$	(8) Reasoning similar as for 7
(9) $\frac{x-x_1}{x_2-x}=\frac{y-y_1}{y_2-y}=\frac{p}{q}$	(9) Statements 4, 5, 6, 7, 8 and given
(10) $\frac{x-x_1}{x_2-x}=\frac{p}{q}$ $\rightarrow qx-qx_1=px_2-px$ $\rightarrow qx+px=px_2+px_1$ $\rightarrow (p+q)\ x=(px_2+qx_1)$ $\rightarrow x=\frac{px_2+qx_1}{p+q}$	(10) Statement 9
(11) $\frac{y-y_1}{y_2-y}=\frac{p}{q}$ $\rightarrow qy-qy_1=py_2-py$ $\rightarrow qy+py=py_2+qy_1$ $\rightarrow (q+p)\ y=py_2+qy_1$ $\rightarrow \quad y=\frac{py_2+qy_1}{p+q}$	(11) Statement 9

Hence the co-ordinates of the point C which divides $\overline{AB}$ in the ratio $p : q$ are $\left(\frac{px_2+qx_1}{p+q}, \frac{py_2+qy_1}{p+q}\right)$.

If C is the mid point of $\overline{AB}$ then $p : q$ becomes 1 : 1 and the co-ordinates of C will be $\left(\frac{x_2+x_1}{1+1}, \frac{y_2+y_1}{1+1}\right)$ or $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$.

Example 8. Find the co-ordinates of a point C which divides A (−10, −11) and B(3, 2) in the ratio 6 : 7.

Solution.

Let the co-ordinates of C be (x, y)

then $x=\dfrac{px_2+qx_1}{p+q}$ and $y=\dfrac{py_2+qy_1}{p+q}$

where $p : q=6 : 7$,

$x_1=-10$, $x_2=3$; $y_1=-11$ and $y_2=2$

$$\therefore \quad x=\frac{6\times(3)+7\times(-10)}{6+7}=\frac{-52}{13}=-4$$

$$y=\frac{6\times(2)+7(-11)}{6+7}=\frac{-65}{13}=-5$$

Hence the co-ordinates of the point C are $(-4, -5)$

Example 9. In what ratio does the point $(1, -1)$ divides the line segment $\overline{AB}$ whose end points are $A(3, -5)$ and $B(-5, 11)$.

Solution.

Let the ratio be $p : q$

We know that $x=\dfrac{px_2+qx_1}{p+q}$, and $y=\dfrac{py_2+qy_1}{p+q}$

In the problem $x_1=3$ $x_2=-5$ and $x=1$

$y_1=-5$ $y_2=11$ $y=-1$

$\therefore$ $1=\dfrac{-5p+3q}{p+q}$ and $-1=\dfrac{11p-5q}{p+q}$

$\rightarrow p+q=-5p+3q$

$\rightarrow 6p=2q$

$\rightarrow p=\frac{1}{3}q$

$\rightarrow \dfrac{p}{q}=\dfrac{1}{3}$

$\therefore p : q=1 : 3$

$\rightarrow -p-q=11p-5q$

$\rightarrow 12p=4q$

$\rightarrow p=\frac{1}{3}q$

$\rightarrow \dfrac{p}{q}=\dfrac{1}{3}$

$\therefore p : q=1 : 3$

Hence the ratio is 1 : 3

Example 10. In what ratio does Y-axis divides the line segment $\overline{AB}$, whose end points are $A(-7, 3)$ and $B(5, -11)$.

Solution.

Let C be the point in common between $\overline{AB}$ and y-axis.

Now X-co-ordinate of C would be O.

∴ Let the co-ordinates of C be $(0, y)$

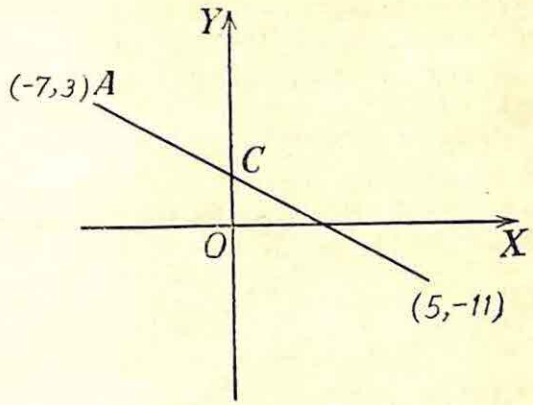


Fig. 229.

Let the required ratio be $p : q$, then

$$0=\frac{px_2+qx_1}{p+q} \quad \text{and} \quad y=\frac{py_2+qy_1}{p+q}$$

$$0=\frac{5p-7q}{p+q} \quad \text{and} \quad y=\frac{-11p+3q}{p+q} \qquad \ldots(1)$$

$$0=\frac{5p-7q}{p+q} \quad \rightarrow 0=5p-7q$$

$$\rightarrow 5p=7q$$

$$\rightarrow \frac{p}{q}=\frac{7}{5}$$

∴ Required ratio is 7 : 5

Now let $p=7k$, then $q=5k$

∴ From (1)

$$y=\frac{-11p+3q}{p+q}$$

$$\rightarrow y=\frac{-77k+15k}{7k+5k}$$

$$\rightarrow y=\frac{-62k}{12k}$$

$$\rightarrow y=\frac{-31}{6}$$

Hence the required ratio is 7 : 5 and the co-ordinates of the common point would be $\left(0, \frac{-31}{6}\right)$.

Example 11. Prove that the line segments $\overline{AB}$ and $\overline{CD}$ bisect each other when the co-ordinates of A, B, C, D are A(7, −9) ; B(3, 5) ; C(0, 2) ; D(10, 2).

Hint : We have to prove that the co-ordinates of mid point of $\overline{AB}$ and that of the mid point of $\overline{CD}$ are the same.

Theorem XIV (4). In a triangle, the medians are concurrent. Also the point of concurrency is the point of trisection of each median. (This we have already proved as a theorem in synthetic geometry).

Given : A $\triangle$ABC, the co-ordinates of whose vertices are $A(x_1, y_1)$; $B(x_2, y_2)$ and $C(x_3, y_3)$.

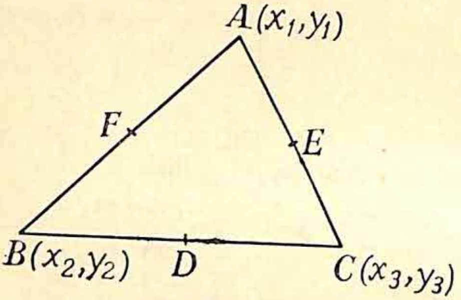


Fig. 230.

D, E, F are the mid points of $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively.

To prove that : $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ have a point in common and that point is the point of trisection of each median.

Proof. (1) the co-ordinates of D, E, F are

$$D\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right);\ E\left(\frac{x_3+x_1}{2}, \frac{y_3+y_1}{2}\right)$$

and $$F\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

The co-ordinates of point G which divides $\overline{AD}$ in the ratio 2 : 1 would be (x, y) such that

$$x=\frac{\frac{2(x_2+x_3)}{2}+1(x_1)}{2+1}=\frac{x_1+x_2+x_3}{3}$$

$$y=\frac{\frac{2(y_2+y_3)}{2}+1(y_1)}{2+1}=\frac{y_1+y_2+y_3}{3}.$$

Similarly if we find the point of trisection of each of medians $\overline{BE}$ and $\overline{CF}$, we will find the co-ordinates of each of these points shall be the same as of G. It implies that $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are concurrent at G. The point of concurrency of medians is called *centroid.*

The co-ordinates of centroid are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

So we notice that we can prove a theorem of Synthetic Geometry with the help of analytical geometry. In fact it is easier to prove some theorems of Euclidean Geometry with the help of analytical method. We further notice that in proving theorems with the help of analytical methods, we make use of Postulates and theorems of Synthetic Geometry. We have stated postulate 22 which connects analytical and synthetic Geometry. However it may be stated that Analytical Geometry can be developed as a system independent of synthetic Geometry. For this purpose we require only two postulates. Postulate 22 and an additional postulate assuming that $d(P_1P_2)$ between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ equals

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

This we have proved with the help of postulates of Synthetic Geometry. However if we assume this as a postulate then we can develop co-ordinate Geometry as a system independent of postulates of Synthetic Geometry. However as our aim in this book is to integrate analytic methods with the methods of Pure Geometry we shall continue to make use of Postulates of Synthetic Geometry.

To find the Area of a Triangular Region in terms of the co-ordinates of the the vertices of the boundary triangle

Theorem XIV (5). The area of a triangular region bounded by a triangle ABC, co-ordinates of whose vertices are $A(x_1y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given by

$$\tfrac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)].$$

Given : A $\triangle$ABC, co-ordinates of its vertices are $A(x_1, y_1)$; $B(x_2, y_2)$ and $C(x_3, y_3)$.

To prove that : The area of the triangular region bounded by $\triangle$ABC will be equal to

$$\tfrac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$

Proof.

Statement	*Reason*
(1) $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are each parallel to Y-axis such that O–D–F–E	(1) Parallel postulate
(2) Area of region bounded by trapezoid ADEB equals to sum of the areas of the region bounded by trap. ADFC, trap. CFEB and $\triangle$ABC	(2) Postulate 20

(3) Area of trapezoid ADEB $=\frac{1}{2}(m\overline{AD}+m\overline{BE})(m\overline{DE})$	(3) Theorem XIII (4)

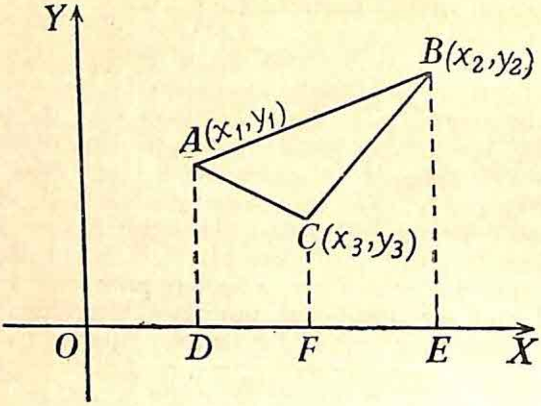


Fig. 231.

(4) Area of trapezoid ADEB $=\frac{1}{2}(y_1+y_2)(x_2-x_1)$	(4) $m\overline{DE}=m(\overline{OE}-m\overline{OD})$ $=x_2-x_1$
(5) Area of trapezoid ADFC $=\frac{1}{2}(y_1+y_3)(x_3-x_1)$	(5) Reason as for 4.
(6) Area of trapezoid CFEB $=\frac{1}{2}(y_2+y_3)(x_2-x_3)$	(6) Reason as for 4
(7) $\frac{1}{2}(y_1+y_2)(x_2-x_1)$ $=\frac{1}{2}(y_1+y_3)(x_3-x_1)$ $+\frac{1}{2}(y_2+y_3)(x_2-x_3)$ +area of region bounded by $\triangle$ABC	(7) Statement 2
(8) Area of region bounded by $\triangle$ABC $=\frac{1}{2}(y_1+y_2)(x_2-x_1)$ $-\frac{1}{2}(y_1+y_3)(x_3-x_1)$ $-\frac{1}{2}(y_2+y_3)(x_2-x_3)$ $=\frac{1}{2}[y_1x_2-y_1x_1+y_2x_2-y_2x_1$ $-y_1x_3+y_1x_1-y_3x_3+y_3x_1$ $-y_2x_2+y_2x_3-y_3x_2+y_3x_3]$ $=\frac{1}{2}[x_1(y_3-y_2)+x_2(y_1-y_3)$ $+x_3(y_2-y_1)]$ $=-\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)$ $+x_3(y_1-y_2)]$.	(8) Statement 7
(9) Area of region bounded by $\triangle$ABC $=\lvert -\frac{1}{2}[x_1(y_2-y_3)$ $+x_2(y_3-y_1)+x_3(y_1-y_2)]\rvert$ $=\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)$ $+x_3(y_1-y_2)]$	(9) Area is a positive number.

It may be noted that we have taken the most general case. Many special cases could have been discussed. Further also it may be noted that the area would always be a positive number. If the vertices A, B, C of a triangle when actually plotted are in clockwise direction then the area of triangular region bounded by △ABC would come out to be negative and then we have to reject the negative sign, since a region bounded by any polygon can be divided into a number of triangles, the formulae proved above can help us in computing the area of any polygon.

Example 10. Find the area of a triangular region bounded by a △ABC, the co-ordinates of whose vertices are A(−2, −1); B(5, −2) and C(4, −4)

Solution. Area of the region bounded by △ABC

$$=\tfrac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$

where $x_1=-2$ $x_2=5$ $x_3=4$

$y_1=-1$ $y_2=-2$ $y_3=-4$

Area of the triangular region

$=\tfrac{1}{2}[-2\{(-2)-(-4)\}+5\{(-4)-(-1)\}+4\{-1)-(-2)\}]$

$=\tfrac{1}{2}[(-4)+(-15)+(4)]=-\tfrac{15}{2}=-7\tfrac{1}{2}.$

Now neglecting the −*ve* sign.

∴ Area of triangular region $=7\tfrac{1}{2}$ square units.

Example 11. Find the area of the region bounded by convex quadrilateral ABCD, the co-ordinates of whose vertices are

A(1, 1) ; B(3,2) ; C(4, 4) ; D(1, 7)

Solution. Area of the triangular region bounded by △ABC

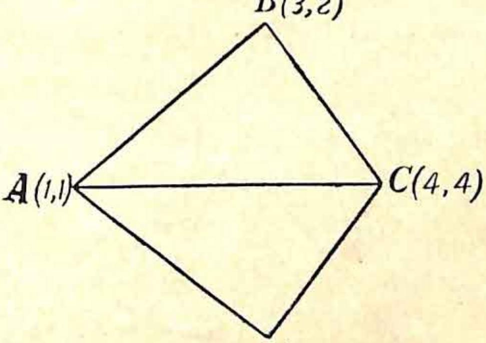


Fig. 232.

$=\tfrac{1}{2}[I(2-4)+3(4-1)+4(1-2)]$

$=\tfrac{1}{2}[-2+9-4]=\tfrac{3}{2}=1\tfrac{1}{2}$ sq. units.

Area of the triangular region bounded by $\triangle ACD$

$$=\tfrac{1}{2}[1(4-7)+4(7-1)+1(1-4)]$$

$$=\tfrac{1}{2}[-3+24-3]=9 \text{ sq. units}$$

$\therefore$ Area of the region bounded by the quadrilateral ABCD

$$=1\tfrac{1}{2}+9=10\tfrac{1}{2} \text{ square units.}$$

Example 12. Show that the points A, B, C whose co-ordinates are A(1, 4) ; B(—3, 16) and C(3, —2) are collinear.

Solution. Assume that A, B, C are not collinear. Then A, B, C will determine a $\triangle ABC$. The area of the region bounded by $\triangle ABC$ will be non-zero. Area of the region bounded by $\triangle ABC$

$$=\tfrac{1}{2}[1\{(16)-(-2)\}+(-3)\{(-2)-(4)\} +3\{(4)-(16)\}]$$

$$=\tfrac{1}{2}[18+18-36]=\tfrac{1}{2}[0]$$

$$=0$$

$\therefore$ A, B, C don't determine a $\triangle ABC$.

Hence A, B, C are collinear.

Theorem XIV (6). Prove that in any triangle, the measure of the line segment joining the middle points of any two sides is half the measure of the third side. (We have already proved it in Synthetic Geometry.)

Given : A $\triangle ABC$, the co-ordinates of whose vertices are

$$A(x_1, y_1)\ ;\ B(x_2, y_2) \text{ and } C(x_3, y_3)$$

F is the mid-point of $\overline{AB}$ and E is the mid point of $\overline{AC}$.

To prove that : $m\overline{EF}=\tfrac{1}{2}m\overline{BC}$.

Proof.

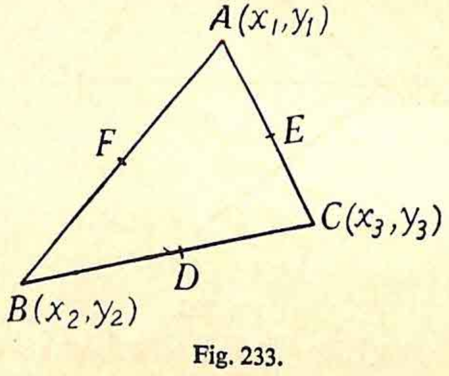


Fig. 233.

Statement	*Reason*
(1) Co-ordinates of F and E are $F\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ and $E\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$	(1) Theorem XIV (2)
(2) $m\overline{EF}$ $=\sqrt{\left(\frac{x_1+x_2}{2}-\frac{x_1+x_3}{2}\right)^2+\left(\frac{y_1+y_2}{2}-\frac{y_1+y_3}{2}\right)^2}$ $=\sqrt{\left(\frac{x_2-x_3}{2}\right)^2+\left(\frac{y_2-y_3}{2}\right)^2}$ $=\frac{1}{2}\sqrt{(x_2-x_3)^2+(y_2-y_3)^2}$	(2) Theorem XIV (1)
(3) $m\overline{BC}$ $=\sqrt{(x_2-x_3)^2+(y_2-y_3)^2}$	(3) Theorem XIV (1)
(4) $m\overline{EF}=\frac{1}{2}m\overline{BC}$	(4) Statements 2, 3.

Theorem XIV (7). Angle bisectors of a triangle are concurrent. (We have also proved it as a theorem in Synthetic Geometry).

Given : A $\triangle ABC$, co-ordinates of whose vertices are

$A(x_1, y_1)$; $B(x_2, y_2)$ and $C(x_3, y_3)$

$m\overline{AB}=c$, $m\overline{BC}=a$

and $m\overline{AC}=b$.

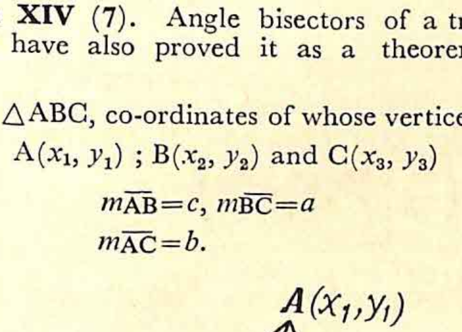


Fig. 234.

To prove that : Angle bisectors of $\angle A$, $\angle B$ and $\angle C$ are concurrent.

Proof. Let $\overline{AD}$ be the angle bisector of $\angle A$ and $\overline{BI}$ be the angle bisector of $\angle B$.

Statement	*Reason*
(1) $\frac{m\overline{BD}}{m\overline{DC}}=\frac{c}{b}$	(1) Theorem 26, $\overline{AD}$ is angle bisector of $\angle A$
(2) D divides $\overline{BC}$; $B(x_2, y_2)$ and $C(x_3, y_3)$ in the ratio $c : b$	(2) Statement 1
(3) Co-ordinates of D are $\frac{cx_3+bx_2}{c+b}, \frac{cy_3+by_2}{c+b}$	(3) Theorem XIV(3)
(4) $\frac{m\overline{DI}}{m\mathrm{IA}}=\frac{m\overline{BD}}{c}=\frac{ac}{b+c}\times\frac{1}{c}$ $=\frac{a}{b+c}$	(4) $\frac{m\overline{BD}}{m\overline{DC}}=\frac{c}{b}$ Statement 1 $\rightarrow \frac{m\overline{BD}}{c}=\frac{m\overline{DC}}{b}=\frac{m\overline{BD}+m\overline{DC}}{c+b}$ $\rightarrow \frac{m\overline{BD}}{c}=\frac{m\overline{DC}}{b}=\frac{a}{b+c}$ $\rightarrow \frac{m\overline{BD}}{c}=\frac{a}{b+c}$ $\rightarrow m\overline{BD}=\frac{ac}{b+c}$
(5) I divides $\overline{AD}$; $A(x_1, y_1)$ and $D\left(\frac{cx_3+bx_2}{b+c}, \frac{cy_3+by_2}{b+c}\right)$ in the ratio $b+c : a$	(5) Statement 4
(6) Co-ordinates of I are $\left[\frac{ax_1+(b+c)\left(\frac{cx_3+bx_2}{b+c}\right)}{a+(b+c)}, \frac{ay_1+(b+c)\left(\frac{cy_3+by_2}{b+c}\right)}{a+(b+c)}\right]$	(6) Theorem XIV (3)
(7) Co-ordinates of I are $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$	(7) Statement 6

Now assume CI′ as the angle bisector of $\angle C$ having I′ a point in common with AD and $I \neq I'$ and compute the co-ordinates of I′ in the same manner as for I

(8) $\frac{mDI'}{mI'A}=\frac{m\overline{CD}}{b}$ $=\frac{ab}{b+c}\times\frac{1}{b}=\frac{a}{b+c}$	(8) As for step 4
(9) I′ divides $\overline{AD}$ in the ratio $b+c : a$	(9) Statement 8
(10) Co-ordinates of I′ are $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$	(10) Theorem XIV (3)
(11) $I \neq I'$ but the co-ordinates of I and I′ are same.	(11) Statements 7 and 10

The statement 11 leads to a contradiction of Postulate 22.

Hence the assumption that $I \neq I'$ has to be abandoned.

Hence the angle bisectors of the angles of a triangle are concurrent. The point of concurrency of the angle bisectors is called the *incentre*. The co-ordinates of incentre of a $\triangle ABC$;

$$A(x_1, y_1)\ ;\ B(x_2, y_2)\ ;\ C(x_3, y_3) \text{ are}$$

$$I\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

Example 13. The co-ordinates of vertices of a $\triangle ABC$ are

$$A(0, -1)\ ;\ B(0, 3)\ ;\ C(7, 2)$$

Find the co-ordinates of its (*a*) centroid G, (*b*) incentre I.

Solution :

Statement	*Reason*
(1) Co-ordinates of G are $G\left(\frac{0+0+7}{3}, \frac{-1+3+2}{3}\right)$ or $G\left(\frac{7}{3}, \frac{4}{3}\right)$	(1) Theorem XIV (4)

(2) $m\overline{BC}=a=\sqrt{(0-7)^2+(3-2)^2}$
$=\sqrt{50}=5\sqrt{2}$
$m\overline{AC}=b=\sqrt{(0-7)^2+(-1-2)}^2$
$=\sqrt{58}$
$m\overline{AB}=c=\sqrt{(0-0)^2+(-1-3)^2}$
$=4$

(2) Theorem XIV (1)

(3) Co-ordinates of I are

$$I\left(\frac{5\sqrt{2}\times 0+\sqrt{58}\times 0+4\times 7}{5\sqrt{2}+\sqrt{58}+4},\ \frac{5\sqrt{2}\times -1+\sqrt{58}\times 3+4\times 2}{5\sqrt{2}+\sqrt{58}+4}\right)$$

$$\text{or } I\left(\frac{28}{5\sqrt{2}+\sqrt{58}+4},\ \frac{-5\sqrt{2}+3\sqrt{58}+8}{5\sqrt{2}+\sqrt{58}+4}\right)$$

(3) Theorem XIV (7)

Problem Set

A

1. Find the co-ordinates of the point which divides the segment $\overline{AB}$, the co-ordinates of whose end points are

   (*a*) A (7,−9) and B (2, 3) in the ratio 1 : 2
   (*b*) A ((−5, 0) and B (5, 0) in the ratio of 2 : 3
   (*c*) A (2, 4) and B (4, 7) in the ratio 5 : 3
   (*d*) A (−3, −4) and B (3, 4) in the ratio 1 : 1
   (*e*) A (−3, 2) and B (4, −3) in the ratio 5 : 3

2. In what ratio does (*i*) X-axis (*ii*) Y-axis divide the line segment $\overline{AB}$, the co-ordinates of whose end points are

   (*a*) A (−5, −9) and B (3, 7)
   (*b*) A (−5, 9) and B (3,−7).
   (*c*) A (5, −9) and B (−3, 7).
   (*d*) A (5, 9) and B (−3, −7).

3. Given A line segment $\overline{AB}$, the co-ordinates of its end points being A (−3, 5) and B (7, −9). There are points C, D, E on $\overline{AB}$ such that A−C−D−E−B and $\overline{AC}\cong\overline{CD}\cong\overline{DE}\cong\overline{EB}$. Find the co-ordinates of C, D and E.

4. Given A quadrilateral ABCD, the co-ordinates of its vertices being A $(-7, 13)$; B $(-3, -6)$: C $(9, -11)$; D $(5, 8)$.

Prove that its diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other.

5. In what ratio will C $(-1, 0)$ divide $\overline{AB}$, the co-ordinates of whose end points are A $(4, 5)$ and B $(-5, -4)$?

B

1. Find the co-ordinates of (*i*) centroid (*ii*) incentre of a $\triangle$ABC, the co-ordinates of whose vertices are :

(*a*) A $(0, 0)$; B $(0, 5)$; C $(5, 0)$.
(*b*) A $(4, 4)$; B $(-4, -4)$ and C $(0, 4)$.
(*c*) A $(-1, 2)$; B $(3, 0)$ and C $(5, 4)$.
(*d*) A $(0, 0)$; B $(a, 0)$ and C $(0, d)$.
(*e*) A $(-36, 7)$; B $(20, 7)$ and C $(0, -8)$.

C

1. Find the area of a triangular region bounded by $\triangle$ABC, the co-ordinates of whose vertices are :

(*a*) A $(0, 0)$; B $(5, 0)$; C $(0, 5)$.
(*b*) A $(4, 4)$; B $(-4, -4)$ and C $(0, 4)$.
(*c*) A $(-1, 2)$; B $(3, 0)$ and C $(5, 4)$.
(*d*) A $(0, 0)$; B $(a, 0)$ and C $(0, d)$.
(*e*) A $(-36, 7)$: B $(20, 7)$ and C $(0, -8)$.
(*f*) A (x, y) ; B (y, z) and C (z, x).
(*g*) A $(3, 4)$; B $(-2, 3)$ and C $(-3, -2)$.

2. Find the area of the region bounded by quadrilateral ABCD, the co-ordinates of whose vertices are .

(*a*) A $(0, 0)$; B $(5, 0)$; C $(0, 5)$; D $(5, 5)$.
(*b*) A $(4, 4)$; B $(-4, -4)$; C $(4, -4)$; D $(-4, 4)$.

3. Show that the points A, B, C whose co-ordinates are as follows, are collinear :

(*a*) A $(6, 6)$; B $(12, 15)$; C $(18, 24)$.
(*b*) A $(1, 4)$; B $(3, -2)$; C $(4, -5)$.

4. Compute the measures of medians of a $\triangle$ABC if the co-ordinates of its vertices are A $(-3, 4)$; B $(7, 9)$ and C $(3, -8)$.

5. Given a quadrilateral ABCD, the co-ordinates of whose vertices are A (x_1, y_1) ; B (x_2, y_2) ; C (x_3, y_3) and D (x_4, y_4). E is the mid point of $\overline{AB}$, F is the mid point of $\overline{BC}$, G is the mid point of $\overline{CD}$ and H is the mid point of $\overline{DA}$. Prove that $\overline{EG}$ and $\overline{FH}$ bisect each other.

CHAPTER XV

THE LOCUS AND ITS EQUATION

We have already talked about Locus in Synthetic Geometry. We defined locus as a geometric figure consisting of a set of points and only those points that satisfy a given condition or a set of conditions. So a curve, which will be a continuous line either curved or straight and which passes through all the points of the set that satisfy the given condition and which does not pass through any other point will be the locus satisfying the given condition. All the points of the plane will be divided into two sets (1) The set consisting of all those points that satisfy the given condition (2) All the points of the plane that does not belong to the set (1) or which does not satisfy the given condition. The set (1) will be the required locus. In co-ordinate geometry, we express the locus in the form of an equation. Suppose we want to find the locus of all those points which are 5 units above the X-axis. It means that Y-co-ordinate of all the points that lie on the required locus will always be 5, although the X-co-ordinate of the different points will be different. So we can write that $y=5$ is the required locus. Clearly the locus will be a line parallel to X-axis. Similarly the locus given by the equation $x=7$ will be a line consisting of all those points which lie at a distance of 7 units from the Y-axis. and parallel to Y-axis. Similarly $x=-10$ will be the set of all those points that are at a distance of 10 units to the left of Y-axis. Clearly the set of points is a line parallel to Y-axis.

The equations $x=y$ and $x=-y$ describe the set of those points which are as far from X-axis as they are from Y-axis. Suppose we have a linear equation $x=y+3$.

This must be true for some ordered pairs of numbers (x, y). We know that this linear equation or statement is true for the ordered pair (7, 4) as we know that $7=4+3$. Similarly we find that this statement is true for ordered pairs (4, 1) ; (5, 2) ; (6, 3) ; (8, 5) ; (9, 6) ; (10, 7)... We can find as many ordered pairs as we want for which this statement would be true. We can draw some co-ordinate axes on a squared or a graph paper and plot or graph these points. On joining all these points, we find that the graph of the equation $x=y+3$ or the locus of all those points that satisfy the condition $x=y+3$ (which simply means that the X-co-ordinate of all the points of the set will always be 3 units more than its Y-co-ordinate) is a line which is not curved or it is what is generally known as a straight line. In case we take any other linear equation of the form $ax+by+c=0$, we will find that its locus will be a line. In fact two fundamental problems of the co-ordinate geometry are :

(1) To find the equation of the curve consisting of the set of all those points that satisfy the given condition or conditions.

(2) To graph or plot the locus of a given equation.

Of course the locus will not always be represented by a linear equation. Later on we will establish that circle is a locus and it is given by the second degree equation $x^2+y^2=r^2$ if the centre of the circle is the origin $(0, 0)$ and its radius is r. Similarly ellipses, parabolas and hyperbolas will be represented by second degree equations of the form $ax^2+by^2=c$; $x^2=ky$ or $y^2=kx$ and $ax^2-by^2=c$ or $xy=c$.

Method to find the equation of locus :

(*i*) Take a general point (x, y) as one of the points of the locus.

(*ii*) Formulate a relation between x and y which should satisfy the given constant (S).

(*iii*) Simplify the relation which will be the equation of the locus.

Example 14. Find the equation of the locus of the set of points such that the distance of each point from X-axis is 5 times its distance from Y-axis.

Solution. Let P (x, y) be any general point on the locus. The distance of P from X-axis is y while the distance of P from Y-axis is x. On applying the given condition, we get that

$$y=5x$$

or $y-5x=0$ which is the equation of the required locus.

Example 15. Find the equation of the locus of the set of points such that each of the points is equidistant from the two given points $(-5, 3)$ and $(7, -5)$.

Solution. Let P (x, y) be any general point on the locus. The distance of P (x, y) from $(-5, 3)$ equals $\sqrt{(x+5)^2+(y-3)^2}$ and the distance of P (x, y) from $(7, -5)$ equals $\sqrt{(x-7)^2+(y+5)^2}$. Now the given condition is that the distance of P (x, y) from each of the points $(-5, 3)$ and $(7, -5)$ is equal.

$$\text{or}\quad \sqrt{(x+5)^2+(y-3)^2}=\sqrt{(x-7)^2+(y+5)^2}$$

$$\leftrightarrow\quad (x+5)^2+(y-3)^2=(x-7)^2+(y+5)^2.$$

$$\leftrightarrow\quad x^2+10x+25+y^2-6y+9=x^2-14x+49+y^2+10y+25.$$

$$\leftrightarrow\quad 10x+14x-6y-10y+25+9-49--25=0.$$

$$\leftrightarrow\quad 24x-16y-40=0.$$

$\leftrightarrow$ $3x-2y-5=0$ which is the equation of the required locus. We have already discussed in synthetic geometry that the locus will be the right bisector of the line segment whose end points are $(-5, 3)$ and $(7, -5)$.

Example 16. Find the locus of the set of points so that the sum of the distances of each of the points from $(c, 0)$ and $(0, c)$ equals k.

Solution. Let P (x, y) be a general point lying on the locus. The distance of P from $(c, 0)$ equals $\sqrt{(x-c)^2+(y-0)^2}$ and its distance from $(0, c)$ equals $\sqrt{(x-0)^2+(y-c)^2}$. Hence on applying the given condition, we get that

$\sqrt{(x-c)^2+y^2}+\sqrt{x^2+(y-c)^2}=k$ is the equation of the required locus.

Problem Set

1. Plot the graph of all those ordered pairs of numbers that make each of the following statements true.

(*a*) $2x+y=8$, (*b*) $5x+3y=16$, (*c*) $3y=21$

(*d*) $4x=-28$ (*e*) $5x=3y$ (*f*) $5x=-3y$

[**Hint.** Consider at least three ordered pairs of numbers that make each of the statements true. Plot the graph of these numbers. The line joining these points will be the required locus.

2. Find the locus of the set of points which satisfy the condition :

(*a*) The distance of each of the points from y-axis is 7 times the distance of the same point from x-axis.

(*b*) The distance of each of the points from $(-3, 7)$ is always 5.

(*c*) The distance of each of the points from the origin $(0, 0)$ is k.

(*d*) Each point is equidistant from $(-7, 5)$ and $(3, -2)$.

(*e*) The sum of distances of each point from the co-ordinate axes is always 7 units.

(*f*) The sum of squares of distances of each point on the locus from the points A (c, d) and B (e, f) equals k.

(*g*) The square of the distance of each point on the locus from C (a, b) equals r^2.

(*h*) The area of triangular region framed by any point on the locus and A $(2, 3)$ and B $(0, 2)$ equals 3.

3. Is a point A $(-3, -1)$ an element of set of points determined by the equation $x^2+y^2-2x-y-17=0$?

[**Hint.** Substitute -3 for x and -1 for y.]

4. Is the point $(3, -7)$ an element of the set of points determined by the equation $4x+3y+9=0$?

5. Is the point (2, 1) an element of set of points determined by the equation $3x^2-4y^2-8=0$?

6. Is the origin (0, 0) an element of set of points determined by the equation $3x^2+y^2-5x+3y=0$?

7. Write an equation that can represent each of the following loci :

(*a*) A line parallel to y-axis and (*i*) three units to the left of it (*ii*) 9 units to the right of it.

(*b*) A line parallel to x-axis and (*i*) 7 units below it (*ii*) 5 units above it.

8. Describe the locus which is represented by the following equations :

(*i*) $x=0$ (*ii*) $y=0$ (*iii*) $x-y=0$ (*iv*) $x+y=0$.

The slope of line

We have already stated that the graph of a linear equation of the form $ax+by+c=0$ is a line. We can find as many ordered pairs of numbers that lie on the locus represented by this equation as we want to it and we can plot the graph of those ordered pairs. On joining the points determined by those ordered pairs we will find that the locus is a line which is not curved. Further the line is continuous without having any end points. In co-ordinate geometry we are interested in knowing the direction of a line. A simple and common way of indicating the direction of a line is to state its slope which can be defined as 'vertical increase per unit of horizontal increase.' In other words, the slope of a line is the measure of increase of a line in vertical direction for every unit increase in horizontal direction.

Suppose there is a line p and on the x-axis the point A represents 2 units and the point B represents 3 units. At A, $\overline{AP}$ is perpendicular to x-axis having a point P in common with line p and at B, $\overline{BQ}$ is perpendicular to x-axis having a point Q in common with line p.

Through P, $\overrightarrow{PS}$ is a ray parallel to x-axis having a point T in common with $\overline{BQ}$. Now the measure of $\overline{QT}$ is the vertical increase for a unit increase of the given line in the horizontal direction. The measure of $\overline{QT}$ gives the slope of the line p.

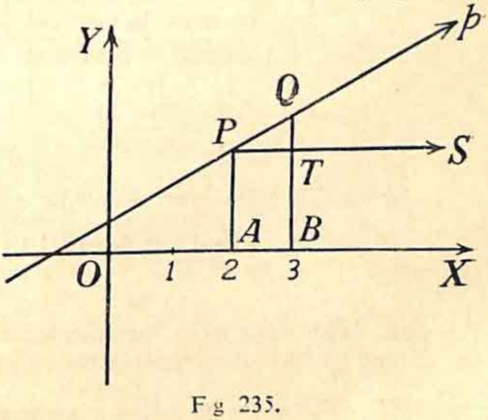


Fig 235.

The x-axis and all the lines parallel to the x-axis are called horizontal lines as these lines do not have any increase in the vertical direction. So we can say that the slope of x-axis and all the lines parallel to x-axis is zero. Y-axis and the lines parallel to Y-axis are called vertical lines. This set of lines have only vertical increase and no horizontal increase. So we find ourselves in a difficult situation when we try to define the slope of this set of lines. In fact this set of lines does not fit into our definition of the slope. So it would be appropriate to talk about the slope of non-vertical lines only. Of course we shall say that the slope of vertical lines does not have any meaning as the division by 'zero' in the system of rational numbers is undefined.

The slope of a line is generally denoted by 'm'. If there is a line p and if there are two points P (x_1, y_1) and Q (x_2, y_2) on the given line with reference to any rectangular axes, then the slope of this line will be given by

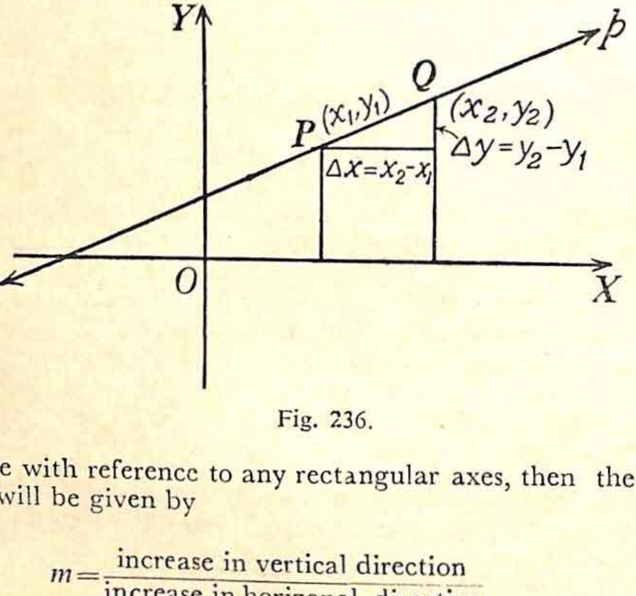


Fig. 236.

$$m = \frac{\text{increase in vertical direction}}{\text{increase in horizonal direction}}$$

$$= \frac{\triangle y}{\triangle x} = \frac{y_2 - y_1}{x_2 -- x_1}$$

From this discussion it will be evident that

(*i*) The lines that are parallel to each other will have the same slope.

(*ii*) The slope of a line is independent of pair of points that we choose on the line to determine the slope.

(*iii*) The slope of a line is same every where.

(*iv*) $m=\frac{y_2-y_1}{x_2-x_1}=\frac{y_1-y_2}{x_1-x_2}$

(*v*) If $m=0$ then $y_1=y_2$ and the line is horizontal and conversely.

(*vi*) If $m>0$ then both the numerator and denominator are either positive or negative and the line rises as we proceed from left to right as is the case with the line p and conversely.

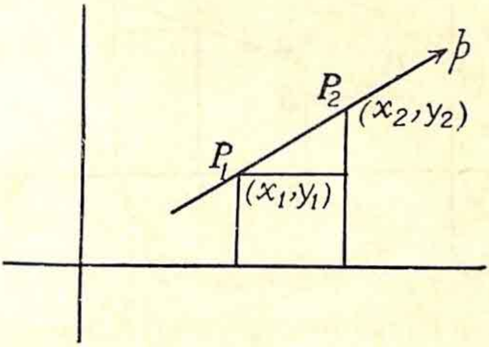


Fig. 237.

(*vii*) If $m<0$ then the denominator is negative if numerator is +ve or numerator is −ve if denominator is +ve and the line falls as we proceed from left to right as is the case with line q and conversely. In this case $m=\frac{y_2-y_1}{x_2-x_1}$ and clearly x_2-x_1 is negative. Let us prove some of these as theorems.

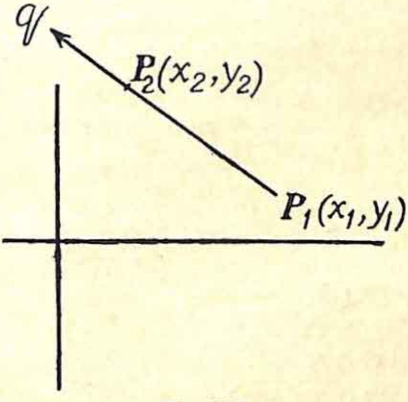


Fig. 238.

Theorem XIV (8). All segments on a non-vertical line have the same slope.

Case First. *When the line is horizontal* :—In this case the line does not have any vertical increase anywhere. So slope of all segments on it is zero.

Case Second. If the line rises as we proceed from left to right.

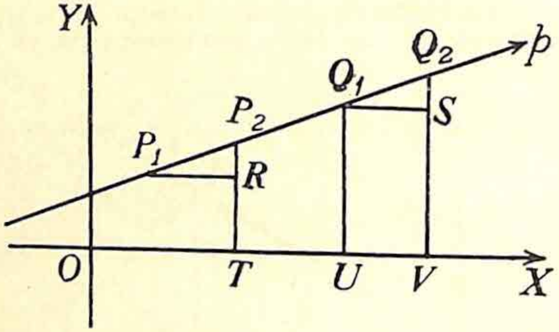


Fig. 239.

Given : A line p and two segments $\overline{P_1 P_2}$ and $\overline{Q_1 Q_2}$.

To prove that :

$$\frac{m\,\overline{P_2R}}{m\,\overline{P_1R}}=\frac{m\,\overline{Q_2S}}{m\,\overline{Q_1S}} \text{ (The slopes of two segments are equal).}$$

Proof.

Statement	*Reason*
(1) $\angle P_1 \cong \angle Q_1$	(1) Why ? Corresponding $\angle s$? How ?
(2) $\angle R \cong \angle S$	(2) Each is a right angle
(3) $\angle P_1P_2R \cong \angle Q_1Q_2S$	(3) Why ?
(4) $1-1$ correspondence $P_1P_2R \leftrightarrow Q_1Q_2S$ is a similarity and $\triangle P_2P_2R \sim \triangle Q_1Q_2S$	(4) Why ?
(5) $\frac{m\,\overline{Q_1Q_2}}{m\,\overline{P_1P_2}}=\frac{m\,\overline{Q_2S}}{m\,\overline{P_2R}}=\frac{m\,\overline{Q_1S}}{m\,\overline{P_1R}}$	(5) Statement 4
(6) $\frac{m\,\overline{Q_2S}}{m\,\overline{P_2R}}=\frac{m\,\overline{Q_1S}}{m\,\overline{P_1R}}$	(6) Statement 5
(7) $\frac{m\,\overline{Q_2S}}{m\,\overline{Q_1S}}=\frac{m\,\overline{P_2R}}{m\,\overline{P_1R}}$	(7) Why ?

Case Third. If the line falls as we proceed from right to left.

Proof.

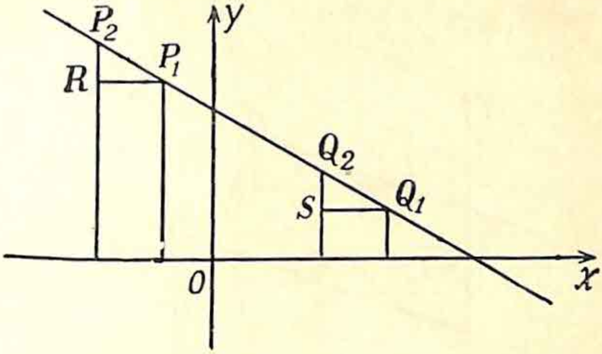


Fig. 240.

Statement	*Reason*
(1) 1—1 correspondence $P_1P_2R \leftrightarrow Q_1Q_2S$ is a similarity and $\triangle P_1P_2R \sim \triangle Q_1Q_2S$	(1) Why ?
(2) $\frac{m\ \overline{P_2R}}{m\ \overline{P_1R}} = \frac{m\ \overline{Q_2S}}{m\ \overline{Q_1S}}$	(2) How ?

In this case both the slopes will be negative as the denominators of both these fractions will be negative.

Theorem XIV (9). Two non-vertical lines are parallel iff (if and only if) they have the same slope. (Iff implies that we have to prove the theorem as well as the converse).

Given : Two lines p and q such that $p \parallel q$. $\overline{PK}$ is a segment on p and $\overline{UV}$ is a segment on q. $\overline{KR}$ $\parallel$ to y-axis and $\overline{PR}$ $\parallel$ to x-axis. $\overline{VW}$ $\parallel$ to y-axis and UW $\parallel$ to x-axis.

To prove that :

$$\frac{m\ \overline{KR}}{m\ \overline{PR}} = \frac{m\ \overline{VW}}{m\ \overline{UW}}$$

Proof.

Statement	*Reason*
(1) $\angle P \cong \angle U$	(1) A problem in chapter on parallel lines in a plane ?
(2) $\angle R \cong \angle W$	(2) Each is a right angle
(3) $\angle PKR \cong \angle UVW$	(3) Why ?

(4) 1 – 1 correspondence PKR↔UVW is a similarity and $\triangle PKR \sim \triangle UVW$ (4) Why ?

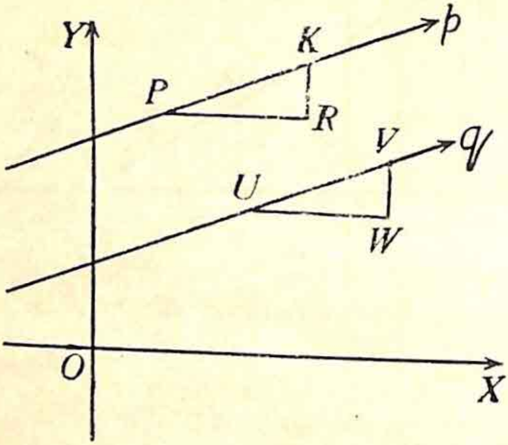


Fig. 241.

(5) $\frac{m\,\overline{KR}}{m\,\overline{PR}} = \frac{m\,\overline{VW}}{m\,\overline{UW}}$ (5) How ?

Conversely.

Given : Two lines p and q. A segment $\overline{PQ}$ on p and a segment $\overline{UV}$ on q. $\overline{QR} \parallel y$-axis ; $\overline{PR} \parallel x$-axis ; $\overline{VW} \parallel y$-axis and $\overline{UW} \parallel x$-axis.

Also $\frac{m\,\overline{QR}}{m\,\overline{PR}} = \frac{m\,\overline{VW}}{m\,\overline{UW}}$

To prove that : $p \parallel q$.

Proof. Assume $p \nparallel q$.

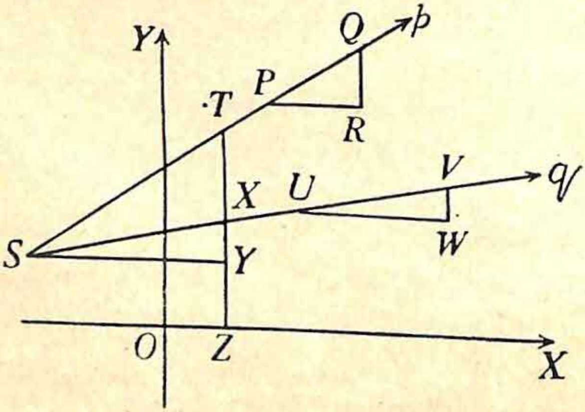


Fig. 242.

Statement	*Reason*
(1) p and q have a common point S	(1) Parallel postulate and the lines are coplanar
(2) There is a point T on p	(2) Which postulate ?
(3) $\overline{TZ} \parallel y$-axis	(3) Parallel postulate
(4) $\overline{TZ}$ has a point X in common with q	(4) $\overline{TZ} \nparallel q$ as q is non vertical and hence not parallel to y-axis
(5) $\overline{SY} \parallel x$-axis having a point Y in common with $\overline{TZ}$	(5) Parallel postulate $\overline{SY} \nparallel \overline{TZ}$ why ?
(6) $\dfrac{m\,\overline{TY}}{m\,\overline{SY}}=\dfrac{m\,\overline{QR}}{m\,\overline{PR}}$	(6) Theorem XIV (8)
(7) $\dfrac{m\,\overline{XY}}{m\,\overline{SY}}=\dfrac{m\,\overline{VW}}{m\,\overline{UW}}$	(7) Theorem XIV (8)
(8) $\dfrac{m\,\overline{QR}}{m\,\overline{PR}}=\dfrac{m\,\overline{VW}}{m\,\overline{UW}}$	(8) Given
(9) $\dfrac{m\,\overline{TY}}{m\,\overline{SY}}=\dfrac{m\,\overline{XY}}{m\,\overline{SY}}$	(9) Statements 6, 7, 8
(10) $m\,\overline{TY}=m\,\overline{XY}$	(10) Statement 9.

Now $m\,\overline{TY}$ can not be equal to m XY as otherwise X=Y and then p, q will have two points S and X or Y in common P which will lead to $p=q$ and then p and q will not be distinct lines.

Therefore $m\,\overline{TY}=m\,\overline{XY}$ leads us to a contradiction. Hence the assumption that $p \nparallel q$ has to be abandoned.

Theorem XIV (10). Two non-vertical lines are perpendicular to each other iff their slopes are negative reciprocals of each other.

Given : Two non-vertical lines p and q such that $p \perp q$. m is the slope of p and m' is the slope of q.

To prove that :

$$m=-\frac{1}{m'}$$

Proof.

Statement	*Reason*
(1) p and q have a point S in common	(1) $p \nparallel q$ and p, q are coplanar

(2) There is a point P on p and a point Q on q such that P is to the right of S and Q is to the left of S and both P and Q lie above S. Also $\overline{SP} \cong \overline{SQ}$	(2) Why ? Segment construction theorem ? Ruler Postulate ? Point plotting theorem ?

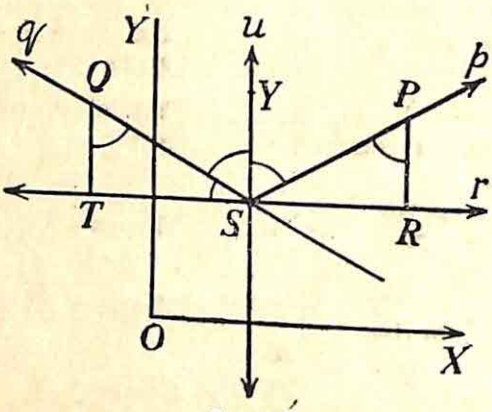


Fig. 243.

(3) Through S, r is a line $\parallel$ x-axis and u is a line $\parallel$ y-axis	(3) Parallel postulate
(4) $\overline{QT}$ is $\parallel$ y-axis having T in common with r and $\overline{PR}$ $\parallel$ y-axis having R in common with r	(4) $\overline{QT} \nparallel r$ $\overline{PR} \nparallel r$
(5) $\angle T \cong \angle R$	(5) Each is a right angle
(6) $\angle RPS \cong \angle PSY$	(6) Interior alternate angles
(7) $\angle SQT \cong \angle QSY$	(7) Why ?
(8) $m\angle PSY + m\angle QSY = 90$	(8) $p \perp q$ given
(9) $m\angle RPS + m\angle SQT = 90$	(9) Statements 6, 7, 8
(10) $m\angle QST + m\angle SQT = 90$	(10) Why ?
(11) $\angle QST \cong \angle RPS$	(11) Statements 9, 10
(12) $\angle SQT \cong \angle RSP$	(12) Why ?
(13) $\overline{SQ} \cong \overline{SP}$	(13) Why ?

(14) 1−1 correspondence SPR↔QST is a congruence and $\triangle SPR \cong \triangle QST$	(14) A.S.A. Theorem
(15) $\overline{SR} \cong \overline{QT}$ and $\overline{PR} \cong \overline{ST}$	(15) Corresponding parts of congruent triangles
(16) $\frac{m\,\overline{PR}}{m\,\overline{SR}} = m$	(16) Given
(17) $m' = -\frac{m\,\overline{QT}}{m\,\overline{ST}}$ $= -\frac{m\,\overline{SR}}{m\,\overline{PR}}$ $= -\frac{1}{m}$	(17) Q is to the left of S

Conversely. Two non-vertical lines p, q ; m is the slope of p and m' is the slope of q, such that $m = -\frac{1}{m'}$

To prove that : $p \perp q$.

Proof.

Statement	*Reason*
(1) $p \nparallel q$	(1) $m \nparallel m'$ as we are given that $m = -\frac{1}{m'}$, and any number can not be negative reciprocal of it
(2) p, q have a common point S	(2) Statement 1
(3) P is a point on p and Q is a point on q such that P is to the right of S and Q to the left of S . Both P and Q lie above S. Also $\overline{SP} \cong \overline{SQ}$.	(3) Why ?
(4) Through S, r is a line ‖ x-axis and u is a line ‖ y-axis	(4) Why ?

(5) $\overline{QT} \parallel y$-axis and $\overline{PR} \parallel y$-axis	(5) Why ?
(6) $\frac{m\,\overline{PR}}{m\,\overline{RS}}=m$ and $\frac{m\,\overline{QT}}{m\,\overline{TS}}=m'$	(6) Given
(7) $\frac{m\,\overline{PR}}{m\,\overline{RS}}=\frac{m\,\overline{ST}}{m\overline{QT}}$	(7) $m=-\frac{1}{m'}$. As Q is to the left of S, therefore no −ve sign.
(8) $\angle R \cong \angle T$	(8) Each is a right angle
(9) 1−1 correspondence $SPR \leftrightarrow QST$ is a similarity and $\triangle SPR \sim \triangle QST$	(9) S.A.S. similarity
(10) $\frac{m\,\overline{QS}}{m\,\overline{SP}}=\frac{m\,\overline{QT}}{m\,\overline{SR}}=\frac{m\,\overline{ST}}{m\,\overline{PR}}$	(10) Statement 9
(11) $\frac{m\,\overline{QS}}{m\,\overline{SP}}=1$	(11) $\overline{QS} \cong \overline{SP}$
(12) $\overline{QT} \cong \overline{SR}$ and $\overline{ST} \cong \overline{PR}$	(12) Statements 10 and 11
(13) 1−1 correspondence $SPR \leftrightarrow QST$ is a congruence and $\triangle SPR \cong \triangle QST$	(13) S.A.S. Postulate
(14) $\angle QST \cong \angle RPS$ and $\angle SQT \cong \angle PSR$	(14) Corresponding parts of congruent triangles
(15) $\angle SQT \cong \angle QSY$	(15) Interior alternate angles
(16) $\angle PSR \cong \angle QSY$	(16) Statements 14, 15
(17) $\angle RPS \cong \angle PSY$	(17) Interior alternate angles
(18) $m\angle PSR + m\angle RPS = m\angle QSY + m\angle PSY$	(18) Statements 16, 17
(19) $m\angle PSR + m\angle RPS = 90$	(19) How ?
(20) $m\angle QSY + m\angle PSY = 90$	(20) Statements 18, 19
(21) $p \perp q$	(21) Statement 20.

Theorem XIV (11). If two intersecting lines have the same slope then the lines coincide.

Given : Two lines p and q, having A in common. Also the slope of the line p equals the slope of the line q.

To prove that $p=q$.

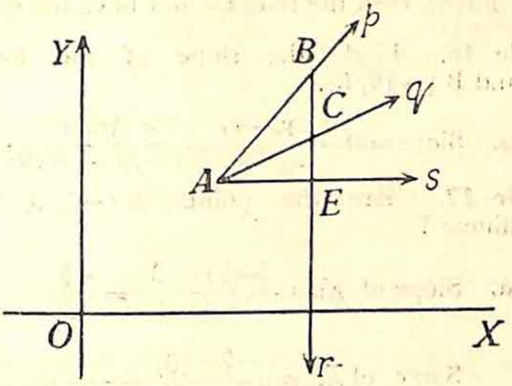


Fig. 244.

Proof. Let (x_1, y_1) be the co-ordinates of A

Statement	*Reason*
(1) There is a point B (x_2, y_2) on p	(1) Why ?
(2) Through B, there is a ray $r \parallel y$-axis	(2) Why ?
(3) r has a point C in common with q and D in common with x-axis	(3) $r \nparallel q$ and $r \nparallel x$-axis
(4) Through A, there is a ray S $\parallel x$-axis having E in common with r	(4) Why ?
(5) The slope of $p=\frac{m\overline{BE}}{m\,\overline{AE}}$ and the slope of $q=\frac{m\,\overline{CE}}{m\,\overline{AE}}$	(5) Definition of slope of a line
(6) $\frac{m\,\overline{BE}}{m\,\overline{AE}}=\frac{m\,\overline{CE}}{m\,\overline{AE}}$	(6) Given
(7) $m\,\overline{BE}=m\,\overline{CE}$	(7) Statement 6
(8) B=C	(8) Statement 7, as B, C, E are on the same line
(9) $p=q$	(9) As p and q have two points A and B in common. Which postulate ?

The contrapositive of the theorem which can be proved on similar lines would be : 'If two intersecting lines have not more than one point in common then the lines do not have the same slope.'

Example 16. Find the slope of the line containing A $(-3, -2)$ and B $(-19, 0)$.

Solution. $\text{Slope} = m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{(-2)-(0)}{(-3)-(-19)} = \dfrac{-2}{16} = \dfrac{-1}{8}$

Example 17. Are the points A $(-2, 3)$; B $(1, -2)$; C $(4, -7)$ collinear ?

Solution. $\text{Slope of } \overline{AB} = \dfrac{(-2)-(3)}{1-(-2)} = \dfrac{-5}{3}$

$$\text{Slope of } \overline{AC} = \frac{-7-(3)}{4-(-2)} = \frac{-10}{6} = \frac{-5}{3}$$

$$\text{Slope of } \overline{BC} = \frac{-7-(-2)}{4-1} = \frac{-5}{3}.$$

As the slope of the segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$ is the same, the points A, B, C are collinear.

Example 18. On a line p, there are points A $(-7, 5)$ and B $(3, -5)$. On another line q, there are points C $(-21, 15)$ and D $(9, -15)$. Is $p \parallel q$?

Solution. $\text{Slope of } p = \dfrac{-5-5}{3-(-7)} = \dfrac{-10}{10} = -1$

$$\text{Slope of } q = \frac{-15-(15)}{9-(-21)} = \frac{-30}{30} = -1.$$

$\therefore$ $p \parallel q$ as their slopes are equal.

Example 19. On a line r, there are points A $(0, 5)$ and B $(1, 12)$. On another line s, there are points C $(-2, 1)$ and D $(-9, 2)$. Is $r \perp S$?

Solution. $\text{The slope of } r = \dfrac{12-5}{1-0} = \dfrac{7}{1} = 7$

$$m' = \text{The slope of } s = \frac{2-1}{-9-(-2)} = \frac{1}{-7} = \frac{-1}{7}$$

as $m' = -\dfrac{1}{m}$, Hence $r \perp s$.

Problem Set

1. For each pair of points given below, find the slope of the line containing both the points.

(*a*) $(0, 0)$ and $(2, 0)$ (*b*) $(3, 4)$ and $(5, 7)$

(*c*) $(-2, 3)$ and $(0, 4)$ (*d*) $(-5, -8)$ and $(-1, -32)$

(*e*) $(9, 3)$ and $(-10, -7)$ (*f*) $(20, 6)$ and $(18, 12)$

2. Find 'a' in the following cases such that the line containing the two points will be horizontal.

(*i*) $(7, a)$ and $(10, 5)$

(*ii*) $(-3, -7)$ and $(-5, a)$.

3. Find 'b' in the following cases such that the line containing the two points will be vertical.

(*i*) $(7, -3)$ and $(b, -7)$ (*ii*) $(b, -11)$ and $(15, -13)$

(*iii*) $(b, 0)$ and $(3, -5)$ (*iv*) $(5, -9)$ and $(b, -10)$

(**Hint.** The x-co-ordinate of the two points should be the same.)

4. Is the point A $(3, 7)$ on the line containing the points B $(2, 8)$ and C $(5, 5)$?

5. Will the line containing $(1, 4)$ and $(2, 3)$ be parallel to the line containing $(0, -6)$ and $(-6, 0)$?

6. Will the line containing $(0, 3)$ and $(-6, 0)$ be perpendicular to the line containing $(0, 3)$ and $(2, -1)$?

7. What is the slope of the line (*i*) parallel to X-axis (*ii*) perpendicular to Y-axis ?

8. What is the equation of X-axis ? Y-axis ?

Linear Equations

We have already discussed that the graph of the solution set for any equation of the form $ax+by=c$ where x and y are the variables and a, b, c are constants is a line. The graph of all the ordered pairs of numbers and only those ordered pairs that satisfy the equation lie on the graph of the line Also each point on the graph of the line corresponds to some ordered pair of numbers that satisfy the given equation. In an advanced course it is possible to prove it. However we shall state it as a postulate.

Postulate 23. The graph of every equation of the form $ax+by=c$ where x and y are the variables and a, b, c are constants (a and b both are not zero) is a line. Also every line is the solution set of some equation of the form $ax+by=c$,

This postulate explains why equations of the form $ax+by=c$ are called linear equations.

Theorem XIV (12). For every linear equation $ax+by=c$ where $b\neq 0$, its slope $m=-\frac{a}{b}$.

Given : A linear equation $ax+by=c$ and $b\neq 0$.

To prove that : The slope of the line is $-\frac{a}{b}$.

Proof.

Statement	*Reason*
(1) There are two points A (x_1, y_1) and B (x_2, y_2) on the line such that $ax_1+by_1=c$ and $ax_2+by_2=c$	(1) Why ?
(2) Slope of the line *i.e.* $m=\frac{y_2-y_1}{x_2-x_1}$	(2) Definition of slope.
(3) $ax_1+by_1=ax_2+by_2$	(3) Statement (1) as $c=ax_1+by_1=ax_2+by_2$
(4) $ax_1-ax_2=by_2-by_1$ $\rightarrow -a(x_2-x_1)=b(y_2-y_1)$ $\rightarrow -\frac{a}{b}=\frac{y_2-y_1}{x_2-x_1}=m$	(4) Statement 3

If $b=0$ then the line becomes $ax=c$ or $x=\frac{c}{a}$ and that means the line is perpendicular to X-axis and it is meaningless to talk about the slope of such a line. The slope of such a line is undefined.

For any linear equation $ax+by=c$ where $b\neq 0$ we have proved that the slope of its graph is $\frac{-a}{b}$. Now we can transform the given equation as follows :

$$ax+by=c$$

$$\rightarrow by=-ax+c$$

$$\rightarrow y=-\frac{a}{b}x+\frac{c}{b}$$

If now we write m for $-\frac{a}{b}$ and k for $\frac{c}{b}$, the equation gets the form $y=mx+k$. So we see that if the linear equation is of the form $y=mx+k$, then the slope of the line is given by the co-efficient of x. Moreover k is the y-intercept *i.e.*, k is the y-co-ordinate of the intersection of the line with Y-axis.

In this connection we can prove the following theorem :

Theorem XIV (13). The equation of a line whose slope is m and whose y-intercept is c is given by $y=mx+c$ if P (x, y) is any general point on the line.

Given : A line p whose slope is m having a common point A with Y-axis such that $m\ \overline{\text{OA}}$ (where O is the origin) is c ($\overline{\text{OA}}$ is the Y-intercept).

To prove that : The equation of the line is $y=mx+c$.

Proof.

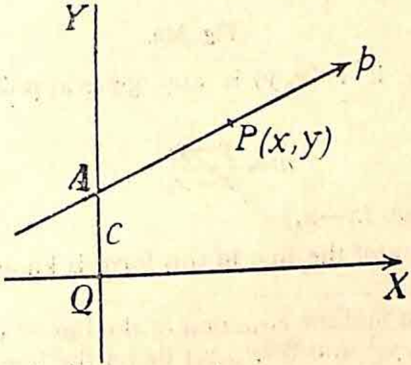


Fig. 245.

Statement	*Reason*
(1) Co-ordinates of A are (0, c)	(1) As A lies on Y-axis, its x-co-ordinate is zero and y-co-ordinate equals $m\overline{\text{OA}}$
(2) Slope of $p=\dfrac{y-c}{x-0}=\dfrac{y-c}{x}$	(2) Definition of slope
(3) $\dfrac{y-c}{x}=m$	(3) Given that the slope of the line is m
(4) $y-c=mx$ $\rightarrow y=mx+c$	(4) Statement 3.

The equation of the form $y=mx+c$ is known as slope intercept form. The equation of a line through origin is $y=mx$ as $c=0$.

In case we are given that the line passes through a given point B (x_1, y_1) and its slope is m then we can prove that the equation of the line is given by

$$y-y_1=m\,(x-x_1)$$

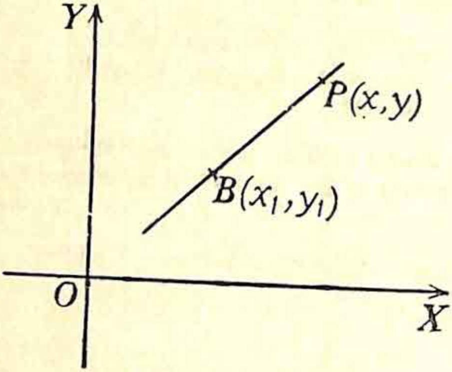


Fig. 246.

In this case if P (x, y) is any general point on the line then its slope

$$m=\frac{y-y_1}{x-x_1}$$

or $y-y_1=m\,(x-x_1)$

The equation of the line in this form is known as **Point Slope Form.**

We can also find the equation of the line if we are given that two points A (x_1, y_1) and B (x_2, y_2) lie on the line.

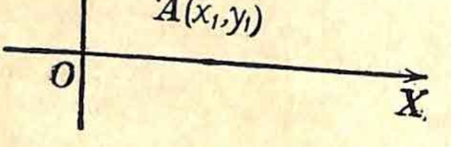


Fig. 247.

As in this case if P (x, y) is any general point on the line, then the slope of the segment $\overline{PA}$ is $\frac{y-y_1}{x-x_1}$ and the slope of the segment $\overline{AB}$ is given by $\frac{y_2-y_1}{x_2-x_1}$.

As we know that segments $\overline{PA}$ and $\overline{AB}$ are on the same line, therefore the slopes are the same.

$\therefore$ $\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$ is the equation of the line passing through A (x_1, y_1) and B (x_2, y_2).

The equation of the line in this form is known as **Two-point form.**

The intercept form. If we are given that a line cuts off an intercept 'a' on X-axis and an intercept 'b' on Y-axis then it implies that we are given two points on the line whose co-ordinates are $(a, 0)$ and $(0, b)$.

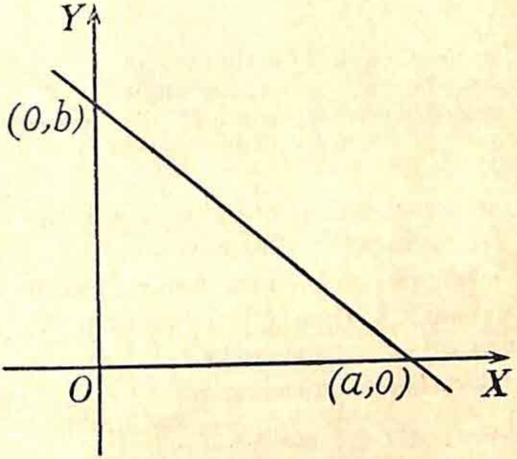


Fig. 248.

So the two-point form equation of the line

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

becomes

$$\frac{y-0}{x-a}=\frac{b-0}{0-a}$$

$$\rightarrow \frac{y}{x-a}=\frac{b}{-a} \rightarrow -ay=bx-ab$$

$$\rightarrow ay+bx=ab \rightarrow \frac{ay}{ab}+\frac{bx}{ab}=\frac{ab}{ab}$$

$$\rightarrow \frac{y}{b}+\frac{x}{a}=1$$

This form is known as *intercept form.*

We can write the equation of a line yet in another form and for that we need to define the following functions :

Sine function. Let S be the set of all right triangles ; Then for each acute angle A of a $\triangle$T belonging to S, consider the ordered pair $\left(m\angle \text{A}, \dfrac{\text{The measure of the side of right } \triangle\text{T opposite A}}{\text{The measure of hypotenuse of } \triangle\text{T}}\right)$.

The set of all such ordered pairs is called the Sine Function and the ratio $\dfrac{\text{the measure of the side of right } \triangle\text{T opposite A}}{\text{the measure of hypotenuse of } \triangle\text{T}}$ is called the Sine of $\angle$A which is also written as sin A.

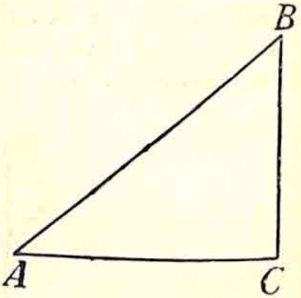


Fig. 249.

If ABC is a right $\triangle$with $\angle$C as right angle, then

$$\text{Sin A} = \frac{m\ (\overline{\text{BC}})}{m\ \overline{\text{AB}}}$$

Cosine Function. Let S be the set of all right triangles, then for each acute angle A of a $\triangle$T belonging to S, consider the ordered pair

$\left(m\angle \text{A}, \dfrac{\text{the measure of the side of right } \triangle\text{T adjacent to A}}{\text{the measure of hypotenuse of } \triangle\text{T}}\right)$. The set of all such ordered pairs is called the cosine function and the ratio $\dfrac{\text{the measure of the side of right } \triangle\text{T adjacent to A}}{\text{the measure of hypotenuse of } \triangle\text{T}}$ is called the cosine of $\angle$A which is also written as cos A.

In the above right $\triangle$, $\cos \text{A} = \dfrac{m\ \overline{\text{AC}}}{m\ \overline{\text{AB}}}$

The Parametric Form. If we are given that a line contains a point A (x_1, y_1) and it makes an angle θ with X-axis, if a

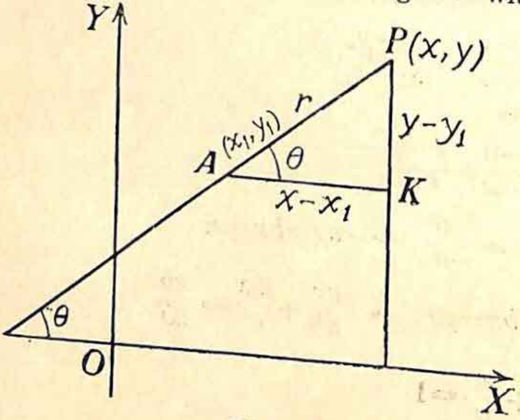


Fig. 250.

general point P (x, y) lies on the line such that $m\,\overline{AP}=r$, then we can prove that the equation of the line is $\dfrac{y-y_1}{\sin\theta}=\dfrac{x-x_1}{\cos\theta}=r$

We can prove it by taking the right △PKA.

Now $m\angle A=\theta$ (How ?) and also

Sin $\theta=\dfrac{y-y_1}{r}$ and $\cos\theta=\dfrac{x-x_1}{r}$

Hence $r=\dfrac{y-y_1}{\sin\theta}$ and $r=\dfrac{x-x_1}{\cos\theta}$

Hence $\dfrac{y-y_1}{\sin\theta}=\dfrac{x-x_1}{\cos\theta}=r$ is the equation of the line.

This form of the equation of the line is called the *Parametric Form*.

The Perpendicular or the Normal Form :

In case we are given the length of the perpendicular segment p, from the origin to a line and also the angle α that the perpendicular segment makes with X-axis, we can prove that the equation of the line would be $x\cos\alpha+y\sin\alpha=y$.

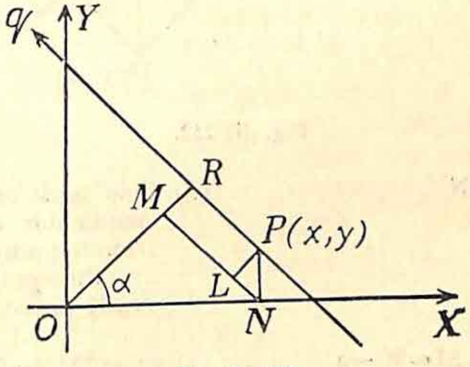


Fig. (i) 251.

Given : A line q, the length of the perpendicular segment OR on q equals p where $R\in q$ and $m\angle ROX=\alpha$ Also P (x, y) is any general point on the line

To prove that : The equation of the line q is given by $x\cos a+y\sin\alpha=p$.

Proof.

Statement	*Reason*
(1) $\overline{PN}\perp\overrightarrow{ON}$ such that $N\in\overrightarrow{OX}$	(1) One and only one perpendicular can be drawn from a point ∉ line to it (A theorem we have already proved)

(2) $\overline{NM} \perp \overline{OR}$ such that $M \in \overline{OR}$	(2) One and only one perpendicular can be drawn from a point ∉ line to it (A theorem we have already proved)

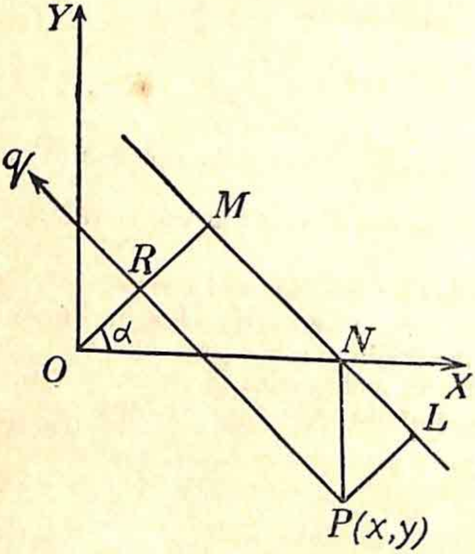


Fig. (ii) 252.

(3) $\overline{PL} \perp MN$	(3) One and only one perpendicular can be drawn from a point ∉ line to it (A theorem we have already proved)
(4) $m\angle ONM = 90 - \alpha$	(4) $m\angle OMN = 90$ and $m\angle MON = \alpha$
(5) $m\angle PNL = \alpha$	(5) Why ? In case of each figure ?
(6) $p = m\,\overline{OR} = m\overline{OM} + m\,\overline{MR}$ [$p = m\overline{OR} = m\overline{OM} - m\overline{MR}$ in the case of fig. (ii)]	(6) Why ?
(7) $\frac{m\overline{OM}}{m\overline{ON}} = \cos\alpha$ and $\frac{m\,\overline{PL}}{m\,\overline{PN}} = \sin\alpha$	(7) By considering the right △OMN and the right △PLN

(8) $\frac{m\,\overline{OM}}{x} = \cos\alpha$ and $\frac{m\,\overline{PL}}{y} = \sin\alpha$ (In the case of figure *ii* it will be $\frac{m\,\overline{OM}}{x} = \cos\alpha$ and $\frac{m\,\overline{PL}}{-y} = \sin\alpha$)	(8) Statement 7 as $m\,\overline{ON}=x$ and $m\,\overline{PN}=y$
(9) $m\overline{OM}=x\cos\alpha$ and $m\overline{PL}=y\sin\alpha$ (or $-y\sin\alpha$ in the case of Fig. *ii*)	(9) Statement 8
(10) $\overline{RM}\cong\overline{PL}$	(10) Opposite sides of a $\parallel^{gm}$
(11) $m\overline{RM}=y\sin\alpha$ or $-y\sin\alpha$	(11) Statements 9 and 10
(12) $p=x\cos\alpha+y\sin\alpha$	(12) Statements 6, 9, 11.

In such a form of equation, sum of squares of co efficients of x and y is 1, as $\cos^2\alpha+\sin^2\alpha=1$ always. (How ?)

So far we have discussed that linear equation of the form $ax+by=c$ is a line. We have also proved that any one of the following forms of an equation will represent a line.

(1) Slope intercept form $y=mx+c$

(2) Intercept form $\frac{y}{b}+\frac{x}{a}=1$

(3) Two point form

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

(4) Normal form

$$x\cos\alpha+y\sin\alpha=p.$$

We can transform any of these equations into general form of linear equation *viz.* $ax+by=c$. This will be as follows :

(1) $y=mx+c \leftrightarrow y-mx=c$ and here $b=1$ and $a=-m$

(2) $\frac{y}{b}+\frac{x}{a}=1 \leftrightarrow \frac{1}{b}\,.\,y+\frac{1}{a}\,.\,x=1 \rightarrow ay+bx=ab$

(3) $\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1} \leftrightarrow y-y_1=(x-x_1)\frac{(y_2-y_1)}{(x_2-x_1)}$

$\leftrightarrow y-y_1=\frac{x(y_2-y_1)}{x_2-x_1}-\frac{x_1(y_2-y_1)}{x_2-x_1}$

$\leftrightarrow y-x\,\frac{(y_2-y_1)}{x_2-x_1}=y_1-x_1\,\frac{(y_2-y_1)}{x_2-x_1}$

$\leftrightarrow (x_2-x_1)y-(y_2-y_1)x=y_1(x_2-x_1)-x_1(y_2-y_1).$

(4) $x \cos \alpha + y \sin \alpha = p$ is itself in the form $ax+by=c$.

We can reduce the general form of the linear [equation to any one of these forms which will be as follows :

(1) To transform into slope form :

$$ax+by=c \leftrightarrow by=-ax+c \leftrightarrow y=\left(\frac{-a}{b}\right)x+\frac{c}{b}$$

(2) To transform into intercept form :

$$ax+by=c \leftrightarrow \frac{a}{c}\, x+\frac{b}{c}\, y=1 \leftrightarrow \frac{x}{\frac{c}{a}}+\frac{y}{\frac{c}{b}}=1$$

(3) To transform into normal form :

$ax+by=c$ is to be changed into

$$x \cos \alpha + y \sin \alpha = p$$

or $$\frac{a}{c}\, x+\frac{b}{c}\, y=1$$

is to be changed into

$$\frac{\cos \alpha}{p}\, x+\frac{\sin \alpha}{p}\, y=1$$

and this implies that

$$\frac{\cos \alpha}{p}=\frac{a}{c}$$

and $$\frac{\sin \alpha}{p}=\frac{b}{c}$$

or $$\cos \alpha=\frac{ap}{c}$$

and $$\sin \alpha=\frac{bp}{c}$$

and since $\cos^2\alpha+\sin^2 \alpha=1$

$$\therefore \quad \frac{a^2p^2}{c^2}+\frac{b^2p^2}{c^2}=1$$

or $$p^2(a^2+b^2)=c^2$$

or $$p=\frac{c}{\sqrt{a^2+b^2}}$$

Hence $$\cos \alpha=\frac{a}{\sqrt{a^2+b^2}}$$

and $$\sin \alpha=\frac{b}{\sqrt{a^2+b^2}}$$

Hence the transformation equation is

$$\frac{ax}{\sqrt{a^2+b^2}}+\frac{by}{\sqrt{a^2+b^2}}=\frac{c}{\sqrt{a^2+b^2}}$$

Let us review

(1) The equation of a line parallel to X-axis and at a distance of 'b' units from it is $y=b$.

(2) The equation of a line parallel to Y-axis and at a distance of 'a' units from it is $x=a$

(3) Equation of X-axis is $y=0$

(4) Equation of axis of y is $x=0$

(5) The equation of a line through origin and whose slope is m is given by $y=mx$

(6) The equation of a line in the slope intercept form is

$$y=mx+c$$

where m is the slope and c is the y-intercept.

(7) The equation of a line in the point-slope form is

$$y-y_1=m(x-x_1)$$

where (x_1, y_1) is a point on the line and m is its slope.

(8) The equation of a line in two-point form is

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

where (x_1, y_1) and (x_2, y_2) are two points on the line.

(9) The equation of a line in the intercept form is

$$\frac{x}{a}+\frac{y}{b}=1$$

where a is the length of x-intercept and b is the length of y-intercept.

(10) The equation of a line in perpendicular form is

$$x\cos\alpha+y\sin\alpha=p$$

where p is the length of perpendicular from origin up to it and α is the angle which it makes with axis of X.

Example 20. Find the equation of a line passing through origin and a point (3, 2).

Solution. We can solve this problem by the following methods :

First Method. We know that the equation of a line through origin and with slope is m is $y=mx$.

Here we do not know m, but we know that $(3, 2)$ is a point on the line. Hence $(3, 2)$ is a solution of

$$y=mx$$

or
$$2=3m$$

or
$$m=\tfrac{2}{3}$$

Hence the equation of the line is

$$y=\tfrac{2}{3}x \leftrightarrow 3y=2x \leftrightarrow 3y-2x=0.$$

Second method. We know that the equation of a line through (x_1, y_1) and with slope m is given by

$$y-y_1=m(x-x_1)$$

As we know that $(3, 2)$ is a point on the line. Hence the equation becomes $y-2=m(x-3)$.

Also we know that the line passes through origin $(0, 0)$. Hence $(0, 0)$ is a solution of this equation

$\therefore \quad 0-2=m(0-3) \leftrightarrow -2=-3m \leftrightarrow 2=3m \leftrightarrow m=\tfrac{2}{3}$
and hence the equation is

$$y-2=\tfrac{2}{3}(x-3)$$
$$\leftrightarrow 3y-6=2x-6 \leftrightarrow 3y=2x \leftrightarrow 3y-2x=0.$$

Third method. The equation of a line in two-point form is

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

The two points that lie on the line are $(0, 0)$ and $(3, 2)$

Hence
$$x_1=0,\ y_1=0\ ;$$
$$x_2=3,\ y_2=2$$

$\therefore$ The equation of the line is

$$\frac{y-0}{x-0}=\frac{2-0}{3-0}$$

$$\leftrightarrow \frac{y}{x}=\frac{2}{3} \leftrightarrow 3y=2x \leftrightarrow 3y-2x=0$$

Example 21. Two points $A(3, 4)$ and $B(-3, -8)$ lie on a line. Find its equation in slope form.

Solution. The equation of a line in slope form is

$$y=mx+c$$

Points $(3, 4)$ and $(-3, -8)$ are its solution sets.

Hence $4=3m+c$ and $-8=-3m+c$.

On solving these simultaneously we get

$$c=-2 \text{ and } m=2.$$

Hence the equation is $y=2x-2$.

Example 22. Find the equation of a line containing the points (2, 4) and (−6, −12). Will this line contain the origin ?

Solution. The equation of a line in two point form is

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

In this case $x_1=2, y_1=4$; $x_2=-6$ and $y_2=-12$.

Hence the required equation is

$$\frac{y-4}{x-2}=\frac{(-12)-(4)}{(-6)-(2)} \leftrightarrow \frac{y-4}{x-2}=\frac{-16}{-8}$$

$$\leftrightarrow \quad \frac{y-4}{x-2}=2\leftrightarrow y-4=2\ (x-2)$$

$$\leftrightarrow \quad y-4=2x-4\leftrightarrow y=2x\leftrightarrow y-2x=0.$$

The line will contain the origin if (0, 0) is its solution set which is so in the case of $y-2=0$. Hence the line $y-2x=0$ contains the origin also.

Example 23. The x-intercept and y-intercept of a line are respectively −3 and +7. Find its equation.

Solution. The equation of a line in the intercept form is

$$\frac{x}{a}+\frac{y}{b}=1.$$

In this problem $a=-3$ and $b=+7$.

Hence the required equation is

$$\frac{x}{-3}+\frac{y}{+7}=1\leftrightarrow-21\left(\frac{x}{-3}+\frac{y}{7}\right)=-21$$

$$\leftrightarrow 7x-3y=-21.$$

Example 24. The co-ordinates of vertices of a △ABC are A (−1, 2) ; B (3, 0) and C (5, 4). Find the equation of the median $\overleftrightarrow{AD}$ where D is the mid-point of $\overline{BC}$.

Solution. The co-ordinates of D are $\left(\frac{3+5}{2}, \frac{0+4}{2}\right)$ or (4, 2).

Now we know the co-ordinates of two points on $\overleftrightarrow{AD}$ *viz.*, A (−1, 2) and D (4, 2). Hence we can make use of two-point form equation

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

The required equation is

$$\frac{y-2}{x-(-1)}=\frac{2-2}{4-(-1)}$$

$$\leftrightarrow \frac{y-2}{x+1}=\frac{0}{5} \leftrightarrow y-2=0$$

Example 25. Find the equation of a line, if the length of perpendicular from origin to it is 5 units and makes an angle of 60° with x-axis. Also find the slope of the line.

Solution. We are given the length of perpendicular from the origin to the line and also the angle it makes with x-axis. Hence we will make use of perpendicular form of the equation *viz.*, $x \cos \alpha + y \sin \alpha = p$.

We know that $p=5$, $\sin \alpha=\sin 60=\frac{\sqrt{3}}{2}$

and $\cos \alpha=\cos 60=\frac{1}{2}$

Hence the required equation is $\frac{1}{2}x+\frac{\sqrt{3}}{2}y=5$

$$\leftrightarrow x+\sqrt{3}\,y=10.$$

Now to find the slope of the line, we will have to transform this equation with slope form *viz.* $y=mx+c$.

$\therefore$ The equation $x+\sqrt{3}\,y=10$ becomes

$$\sqrt{3}y=-x+10 \leftrightarrow y=-\frac{1}{\sqrt{3}}x+\frac{10}{\sqrt{3}}$$

$\therefore$ The slope of the line is $\frac{--1}{\sqrt{3}}$

Example 26. Find the length of the perpendicular from the origin to $2x-3y=6$ and also the angle it makes with x-axis.

Solution. We have to transform the given equation in the form $x \cos \alpha+y \sin \alpha=p$.

The given equation is

$$2x-3y=6$$

$$\leftrightarrow \frac{2x}{\sqrt{(2)^2+(-3)^2}}-\frac{3y}{\sqrt{(2)^2+(-3)^2}}=\frac{6}{\sqrt{(2)^2+(-3)^2}}$$

$$\leftrightarrow \frac{2x}{\sqrt{13}}-\frac{3y}{\sqrt{13}}=\frac{6}{\sqrt{13}}$$

$$\therefore\ p=\frac{6}{\sqrt{13}},$$

and $\cos\ \alpha=\frac{2}{\sqrt{13}}$,

$\sin\ \alpha=\frac{-3}{\sqrt{13}}$

$\therefore\quad \tan\alpha=-\ \frac{2}{3}$

$\therefore\qquad \alpha=\tan^{-1}-\ \frac{2}{3}$

Problem Set

1. Find the equation of a line :

(*a*) Parallel to x-axis and 5 units above it.

(*b*) Parallel to x-axis and 2 units below it.

(*c*) Parallel to y-axis and at a distance of 4 units from it.

(*d*) Parallel to y-axis and at a distance of 3 units from it.

(*e*) Passing through the origin and whose slope is $-\frac{1}{2}$.

(*f*) Passing through the origin and a point P $(-3, -4)$.

(*g*) Whose slope is $\frac{3}{4}$ and y-intercept is -7.

(*h*) Whose slope is $-\sqrt{3}$ and y-intercept is 5.

(*i*) Which passes through the point P $(-5, 2)$ and whose slope is $\frac{1}{\sqrt{2}}$.

(*j*) Whose slope is $\frac{2}{5}$ and which passes through the point $(-2, 3)$.

(*k*) Which passes through A $(-3, -5)$; B $(7, -3)$.

(*l*) If $p=3$ and $\alpha=30$.

2. Find the x-intercept as well as y-intercept of each of the equations of Problem I.

3. Find the slope of the line given by each of equation of Problem I wherever it is meaningful.

4. Find the slope and x-intercept as well as the y-intercept of each of the following equations

(*a*) $3y+2x=6$ (*b*) $2y-5x=7$.

5. Find p and α in the case of each of equations of Problem IV.

6. Find the equation of a line through $(-2, 3)$ and parallel to the line $4x+5y=3$.

(**Hint.** The slope of the line will be $-\frac{4}{5}$)

7. Find the equation of a line through $(3, -5)$ and perpendicular to the line $5x+2y=4$.

(**Hint.** The slope of the line will be $\frac{2}{5}$; How ?)

The point of intersection of two lines :

Let us consider two lines whose equations are $3x-2y=4$ and $2x+3y=7$.

Since the lines are not parallel as their slopes are different, hence these lines intersect. We can write as many pairs of ordered numbers as we want which will be the solution set of each of these equations.

First take the equation $3x-2y=4$;

Some of the ordered pairs of numbers that are its solution set are

$(0, -2)$; $(4, 4)$; $(2, 1)$; $(6, 7)$; $(8, 10)$...

Similarly for the second equation $2x+3y=7$,

Some of the ordered pairs of numbers that are its solution set are.

$(\frac{7}{2}, 0)$; $(0, \frac{7}{3})$; $(2, 1)$; $(5, -1)$ $(8, -3)$...

We find that the point represented by the ordered pair $(2, 1)$ is on both the lines. Hence this point is the point of intersection of these two lines. Of course we can find the co-ordinates of the point of intersection by solving the two equations simultaneously as follows :

$$3x-2y=4 \text{ and } 2x+3y=7$$

$$\leftrightarrow 2(3x-2y=4)-3(2x+3y=7)$$

$$\leftrightarrow (6x-4y=8)-(6x+9y=21)$$

$$\leftrightarrow 6x-6x-4y-9y=8-21.$$

$$\leftrightarrow -13y=-13 \leftrightarrow y=1$$

Also $y=1$ and $3x-2y=4 \leftrightarrow 3x-2=4$

$\leftrightarrow 3x=6 \leftrightarrow x=2$

Let us consider the general case :

To find the point of intersection of the lines

$$a_1x+b_1y=c_1 \text{ and } a_2x+b_2y=c_2.$$

$$a_1x+b_1y=c_1$$

and $$a_2x+b_2y=c_2$$

$\leftrightarrow\ a_2(a_1x+b_1y=c_1)-a_1(a_2x+b_2y=c_2)$

$\leftrightarrow\ (a_1a_2x+a_2b_1y=a_2c_1)-(a_1a_2x+a_1b_2y=a_1c_2)$

$\leftrightarrow\ a_1a_2x-a_1a_2x+a_2b_1y-a_1b_2y=a_2c_1-a_1c_2$

$\leftrightarrow\ y(a_2b_1-a_1b_2)=a_2c_1-a_1c_2$

$\leftrightarrow\qquad y=\dfrac{a_2c_1-a_1c_2}{a_2b_1-a_1b_2}$

Also $a_1x+b_2y=c_1$, and

$$y=\frac{a_2c_1-a_1c_2}{a_2b_1-a_1b_2}$$

$\leftrightarrow\ a_1x+b_1\dfrac{(a_2c_1-a_1c_2)}{(a_2b_1-a_1b_2)}=c_1$

$\leftrightarrow\qquad a_1x=c_1--\dfrac{b_1(a_2c_1-a_1c_2)}{(a_2b_1-a_1b_2)}$

$\leftrightarrow\qquad a_1x=\dfrac{a_2b_1c_1-a_1b_2c_1-b_1a_2c_1+a_1c_2b_1}{a_2b_1-a_1b_2}$

$\leftrightarrow\qquad a_1x=\dfrac{a_1(b_1c_2-b_2c_1)}{a_2b_1-a_1b_2}$

$\leftrightarrow\qquad x=\dfrac{b_1c_2-b_2c_1}{a_2b_1-a_1b_2}$

Hence the point of intersection is

$$\left(\frac{b_1c_2-b_2c_1}{a_2b_1-a_1b_2},\ \frac{a_2c_1-a_1c_2}{a_2b_1-a_1b_2}\right)$$

Concurrency of three lines when the lines are not parallel. To show that three given lines are concurrent, we must show that point of intersection of any two lines is also the element of the third line.

To show that the lines

$$3x+2y=13$$
$$2x-3y=0$$
$$x+2y=7$$

are concurrent.

We find the point of intersection of the lines

$$3x+2y=13 \text{ and}$$
$$2x-3y=0$$

$\leftrightarrow\ 2(3x+2y=13)-3(2x-3y=0)$

$\leftrightarrow\ (6x+4y=26)-(6x-9y=0)$

$$\leftrightarrow \quad 6x-6x+4y+9y=26-0$$

$$\leftrightarrow \quad 13y=26$$

$$\leftrightarrow \quad y=2$$

Also $$3x+2y=13$$

and $$y=2$$

$$\leftrightarrow \quad 3x+4=13$$

$$\leftrightarrow \quad 3x=13-4 \leftrightarrow 3x=9 \leftrightarrow x=3$$

So the point of intersection is $(3, 2)$.

Now the point $(3, 2)$ is element of the third line as

$x+2y=7$ is satisfied by $(3, 2)$.

$$\because \quad 3+4=7$$

is a true statement.

Hence the three lines are concurrent.

Lines through point of intersection of two lines. Through the intersection of two lines we will have an infinite set of lines.

If we are given the equation of two lines

$$a_1x+b_1y=c_1$$

and $$a_2x+b_2y=c_2$$

then the equation of the line through the intersection of these will be given by

$$(a_1x+b_1y=c_1)+k(a_2x+b_2y=c_2)$$

$$\leftrightarrow \quad x(a_1+ka_2)+y(b_1+kb_2)=c_1+kc_2$$

as the point of intersection of these will be

$$\left(\frac{b_1c_2-c_1b_2}{a_2b_1-a_1b_2}, \frac{a_2c_1-a_1c_2}{a_2b_1-a_1b_2}\right)$$

which lies on

$$x(a_1+ka_2)+y(b_1+kb_2)=c_1+kc_2$$

$$\text{as } \left(\frac{b_1c_2-c_1b_2}{a_2b_1-a_1b_2}\right)\left(a_1+ka_2\right)+\left(\frac{a_2c_1-a_1c_2}{a_2b_1-a_1b_2}\right)\left(b_1+kb_2\right)$$

$$=c_1+kc_2$$

$$\rightarrow \quad \frac{1}{a_2b_1-a_1b_2}[(b_1c_2-c_1b_2)(a_1+ka_2)+(a_2c_1-a_1c_2)(b_1+kb_2)]$$

$$=c_1+kc_2$$

$$\rightarrow \quad \frac{1}{a_2b_1-a_1b_2}[a_1b_1c_2-a_1c_1b_2+kb_1c_2a_2-ka_2c_1b_2+b_1a_2c_1-b_1a_1c_2$$

$$+ka_2c_1b_2-ka_1c_2b_2]$$

$$=c_1+kc_2$$

$$\rightarrow \quad \frac{1}{a_2b_1-a_1b_2}\,[c_1(a_2b_1-a_1b_2)+kc_2(a_2b_1-a_1b_2)]$$
$$=c_1+kc_2$$
$$\rightarrow \quad \frac{(c_1+kc_2)(a_2b_1-a_1b_2)}{(a_2b_1-a_1b_2)}$$
$$=c_1+kc_2$$
$$\rightarrow \quad c_1+kc_2=c_1+kc_2$$

Example 27. Find the equation of the line through the intersection of $3x+2y=8$ and $4x+y=14$ containing $P(7, 9)$.

Solution. A line through the intersection of

$$3x+2y=8 \text{ and}$$
$$4x+y=14 \text{ is given by}$$
$$(3x+2y=8)+k(4x+y=14)$$
$$\rightarrow \quad 3x+4kx+2y+ky=8+14k$$
$$\rightarrow \quad x(3+4k)+y(2+k)=8+14k$$

Now it contains $P(7, 9)$ and hence it will be the solution set of

$$x(3+4k)+y(2+k)=8+14k$$
$$\text{or} \quad 7(3+4k)+9(2+k)=8+14k$$
$$\rightarrow \quad 21+28k+18+9k=8+14k$$
$$\rightarrow \quad 37k+39=8+14k$$
$$\rightarrow \quad 23k=-31$$
$$\rightarrow \quad k=\frac{-31}{23}$$

Hence the required equation is

$$x\left[3+4\left(\frac{-31}{23}\right)\right]+y\left[2+\left(\frac{-31}{23}\right)\right]$$
$$=8+14\left(\frac{-31}{23}\right)$$
$$\rightarrow \quad x\left(\frac{-55}{23}\right)+y\left(\frac{15}{23}\right)$$
$$=\frac{184-434}{23}$$
$$\rightarrow \quad -55x+15y=-250$$
$$\rightarrow \quad 11x-3y=50$$

Alternatively we could have found the point of intersection of the two lines which will be :

$$4(3x+2y=8)-3(4x+3y=14)$$

$$\rightarrow\ (12x+8y=32)-(12x+3y=42)$$

$$\rightarrow\ 12x-12x+8y-3y=32-42$$

$$\rightarrow\ 5y=-10$$

$$\rightarrow\ y=-2$$

Also $\quad 3x+2y=8$ and

$$y=-2$$

$$\leftrightarrow\ 3x-4=8 \leftrightarrow 3x=12 \leftrightarrow x=4$$

Therefore the two points on the line are

$$(7, 9) \text{ and } (4, -2)$$

The two-point form of the equation is

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

Hence the required equation is

$$\frac{y-9}{x-7}=\frac{-2-9}{4-7} \leftrightarrow \frac{y-9}{x-7}=\frac{-11}{-3}$$

$$\leftrightarrow\ \frac{y-9}{x-7}=\frac{11}{3} \leftrightarrow 3y-27=11x-77$$

$$\leftrightarrow\ 11x-3y=50$$

Example 28. Find the equation of a line through the intersection of $4x-3y=5$ and $3x+4y=10$ and

(*a*) Whose slope is $\frac{3}{4}$

(*b*) Which is ∥ to the line $3x+7y=13$

(*c*) Which is perpendicular to the line $2x-5y=7$

(*d*) Its x-intercept equals y-intercept in magnitude.

Solution. Any equation through the intersection of

$$4x-3y=5 \text{ and}$$
$$3x+4y=10$$

is given by

$$(4x-3y=5)+k(3x+4y=10)$$

$$\rightarrow\ x(4+3k)+y(4k-3)=5+10k$$

Now

(*a*) if the slope is $\frac{3}{4}$

The slope form of the equation will be

$$y(4k-3)=-(4+3k)x+(5+10k)$$

$$y=-\frac{(4+3k)}{4k-3}x+\frac{5+10k}{4k-3}$$

Hence the slope of the line is

$$-\frac{(4+3k)}{4k-3}=\frac{3}{4}$$

$$\leftrightarrow \qquad -16-12k=12k-9$$

$$\leftrightarrow \qquad -7=24k \rightarrow k=\frac{-7}{24}$$

Hence the required equation is

$$x\left(4-\frac{7}{8}\right)+y\left(-\frac{7}{6}-3\right)=5-\frac{70}{24}$$

$$\leftrightarrow \qquad \frac{25x}{8}-\frac{25}{6}y=\frac{50}{24}$$

$$\leftrightarrow \qquad \frac{x}{8}-\frac{y}{6}=\frac{1}{12}$$

$$\leftrightarrow \qquad 3x-4y=2$$

(*b*) In this case the slope of the line is to be the same as that of

$$3x+7y=13$$

$$3x+7y=13 \rightarrow 7y=-3x+13 \rightarrow y=\frac{-3}{7}x+\frac{13}{7}$$

$\therefore$ Slope of the line $3x+7y=13$ is $\frac{-3}{7}$

Hence
$$\frac{-(4+3k)}{4k-3}=\frac{-3}{7} \leftrightarrow \frac{4+3k}{4k-3}=\frac{3}{7}$$

$$\leftrightarrow \qquad 28+21k=12k-9$$

$$\leftrightarrow \qquad 21k-12k=-9-28 \leftrightarrow 9k=-37$$

$$\leftrightarrow \qquad k=\frac{-37}{9}$$

$\therefore$ On putting this value of $k=\frac{-37}{9}$ in

$$x(4+3k)+y(4k-3)=5+10k$$

we get the required equation as

$$x\left[4+3\left(\frac{-37}{9}\right)\right]+y\left[4\left(\frac{-37}{9}\right)-3\right]=5+10\left(\frac{-37}{9}\right)$$

$$\leftrightarrow \qquad x\left[\frac{12-37}{3}\right]+y\left(\frac{-148-27}{9}\right)=\frac{45-370}{9}$$

$$\leftrightarrow \qquad x\left(\frac{-25}{3}\right)+y\left(\frac{-175}{9}\right)=\frac{-325}{9}$$

$$\leftrightarrow -\frac{x}{3}-\frac{7y}{9}=-\frac{13}{9}$$

$$\leftrightarrow 3x+7y=13$$

Which also show that the line $3x+7y=13$ passes through the point of intersection of the lines $4x-3y=5$ and $3x+4y=10$.

(*c*) The slope form of $2x-5y=7$ will be

$$5y=2x-7 \leftrightarrow y=\frac{2}{5}x-\frac{7}{5}$$

Hence its slope is $\frac{2}{5}$.

$\therefore$ The slope of the required line will be $\frac{-5}{2}$

$$\therefore \quad \frac{-(4+3k)}{4k-3}=\frac{-5}{2} \leftrightarrow \frac{4+3k}{4k-3}=\frac{5}{2}$$

$$\leftrightarrow 8+6k=20k-15 \leftrightarrow 23=14k.$$

$\leftrightarrow k=\frac{23}{14}$ and on putting this value of k in

$$x(4+3k)+y(4k-3)=5+10k.$$

We get the equation of the required line as

$$x\left[4+3\left(\frac{23}{14}\right)\right]+y\left[4\left(\frac{23}{14}\right)-3\right]=5+10\left(\frac{23}{14}\right)$$

$$\leftrightarrow \quad x\left(\frac{56+69}{14}\right)+y\left(\frac{92-42}{14}\right)=\frac{70+230}{14}$$

$$\leftrightarrow \quad 125x+50y=300$$

$$\leftrightarrow \quad 5x+2y=12.$$

(*d*) The intercept form of the equation

$$x\,(4+3k)+y\,(4k-3)=5+10k \text{ is}$$

$$x\,\frac{(4+3k)}{5+10k}+y\,\frac{(4k-3)}{5+10k}=1$$

$$\leftrightarrow \quad \frac{x}{\frac{5+10k}{4+3k}}+\frac{y}{\frac{5+10k}{5k-3}}=1$$

as x intercept equals y-intercept

$$\therefore \quad \frac{5+10k}{4+3k}=\frac{5+0k\ 1}{4k-3}$$

$$\leftrightarrow \quad (5+10k)\left(\frac{1}{4+3k}-\frac{1}{4k-3}\right)=0$$

$$\leftrightarrow \quad (5+10k)\,\frac{(4k-3-4-3k)}{(4+3k)(4k-3)}=0$$

$$\leftrightarrow \qquad (5+10k)\,(k-7)=0$$

hence $5+10k=0 \quad \leftrightarrow \qquad k=-\frac{1}{2}$

or $k-7=0 \rightarrow k=7$.

We will get two distinct equations on putting

$k=-\dfrac{1}{2}$ and $k=7$ in $x(4+3k)+y(4k-3)=5+10k$.

Which will be

(*i*) when $k=-\dfrac{1}{2}$

The equation is

$$x\left(4-\frac{3}{2}\right)+y(-2-3)=5-5.$$

$$\leftrightarrow \qquad \frac{5x}{2}-5y=0$$

$$\leftrightarrow \qquad 5x-10y=0$$

$$\leftrightarrow \qquad x-2y=0$$

It shows that this line passes through origin and hence both the intercepts are zero in measure.

(*ii*) When $k=7$.

The equation is

$$x(4+21)+y(28-3)=5+70$$

$$\leftrightarrow \qquad 25x+25y=75$$

$$\leftrightarrow \qquad x+y=3$$

Which shows that both the intercepts are 3 units in measure.

Example 29. The equations of lines forming a rectangle ABCD are as follows :

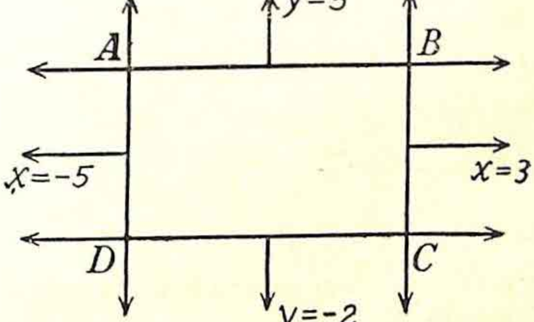


Fig. 253.

(*i*) $x=3$ (*ii*) $x=-5$ (*iii*) $y=5$ (*iv*) $y=-2$.

Find the equations of lines containing the diagonals $\overline{BD}$ and $\overline{AC}$.

Solution. It is clear that $x=3$ and $x=-5$ will be lines ‖ to each and similarly $y=-2$ and $y=5$ are the two lines parallel to each other.

The equation of the line containing $\overline{BD}$ will be given by $(x=3)+k(y=5)$ and also

by $(x=-5)+l(y=-2)$

or $\quad x+ky=3+5k$ and $x+ly=-5-2l$.

Now these two equations represent the same line.

$\therefore$ Co-efficients of x, y and constant terms will be proportional.

or
$$\frac{1}{1}=\frac{k}{l}=\frac{3+5k}{-5-2l}$$

$$\rightarrow \quad k=l \text{ and } 3+5k=-5-2l$$

$$\rightarrow \quad k=l \text{ and } 5k+2l=-8$$

$$\rightarrow \quad 5k+2k=-8$$

$$\rightarrow \quad k=-\frac{8}{7}$$

$$\therefore \quad l=-\frac{8}{7}$$

Hence the equation of the line is

$$x+ky=3+5k$$

$$\leftrightarrow \quad x-\frac{8}{7}y=3+5\left(-\frac{8}{7}\right)$$

$$\leftrightarrow \quad 7x-8y=21-40$$

$$\leftrightarrow \quad 7x-8y=-19.$$

or $\quad$ taking $x+ly=-5-2l$

The equation of the line is

$$x-\frac{8}{7}y=-5+\frac{16}{7}$$

$$7x-8y=-19.$$

Proceeding on similar lines we can find the equation of the line containing the diagonal $\overline{AC}$.

Alternatively. The co-ordinates of A, B, C, D respectively will be $(-5, 5)$; $(3, 5)$; $(3, -2)$; $(-5, -2)$.

Hence the equation of the line containing the diagonal $\overline{BD}$ will be :—

$$\frac{y-5}{x-3}=\frac{(-2)-(5)}{(-5)-(3)} \leftrightarrow \frac{y-5}{x-3}=\frac{-7}{-8}$$

$$\leftrightarrow \quad \frac{y-5}{x-3}=\frac{7}{8} \leftrightarrow 8(y-5)=7(x-3)$$

$$\leftrightarrow \quad 8y-40=7x-21 \leftrightarrow 7x-8y=-19.$$

Problem Set

1. Find the co-ordinates of the point of intersection of

(*a*) $5x+y=11$ and $x-5y=-3$

(*b*) $3x+2y=14$ and $5x-y=19$.

2. Find the area of triangular regions formed by the lines

(*a*) $x-y=1$; $x+y=3$; $2x-y=-2$

(*b*) $x+y=0$; $x-y=0$; $y=4$

(*c*) $3x+4y=-1$; $2x-y=3$; $7y-3x=23$.

3. Show that the sets of lines given below are concurrent. Also find the co-ordinates of the point of concurrency.

(*a*) $3x+2y=19$; $5x-7y=11$ and $x+5y=20$

(*b*) $5x-3y=4$; $3x+4y=-15$ and $2x-7y=19$

(*c*) $7x+3y=7$; $5x-9y=-73$ and $x+2y=12$.

4. For what value of b will the following sets of lines be concurrent ? Also find the co-ordinates of the point of concurrency.

$4x-3y=0$; $-2x+5y=14$ and $5x+by=23$.

5. Find the equation of the line through the intersection of $3x+4y=5$ and $4x-3y=7$ and containing the origin.

6. Find the equation of the line through

$2x-3y=5$ and $3x+2y=14$ and

(*i*) parallel to the line $4x+7y=11$

(*ii*) perpendicular to the line $7x-4y=11$

(*iii*) through the point $(-3, -5)$

(*iv*) having a slope $\frac{1}{2}$

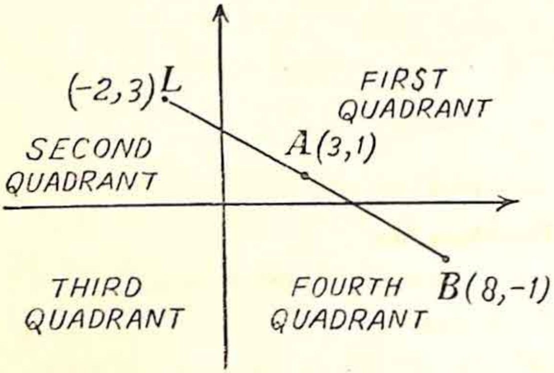
(*v*) having x-intercept and y-intercept each equal to 1.

7. Find the co-ordinates of orthocentre of the triangle formed by the lines $x-y=3$; $x+y=-3$ and $2x+y=9$.

8. Find the equations of diagonals of a quadrilateral formed by the lines $x+y=5$; $x-y=3$; $x+y=-7$ and $x-y=-5$.

Graph of a linear equation in two variables.

We have already talked about how to graph an equation in two variables like $2x+5y=11$. To do so we would choose two reference lines or axes intersecting at right angles in the plane of paper. Their point of intersection is called origin whose coordinates are taken to be (0, 0). Then we choose units of length according to one scale. We also indicate the horizontal axis as X-axis and the vertical axis as Y-axis. We also indicate the four regions or quadrants into which the plane is divided by the axis. Then we take at least three ordered pairs of numbers that are the solution set of the given equation ; $(3, 1)$, $(8, -1)$, $(-2, 3)$ are the solution sets of $2x+5y=11$. Let $A(3, 1)$; $B(8, -1)$ and $L(-2, 3)$ be the points. The abscissa of A is 3 and its ordinate is 1. So the graph of A will be in the first quadrant. Graph of B will be in the fourth quadrant. The graph of L will be in the second quadrant. By joining these points and extending it both ways we get the graph of the line $2x+5y=1$.



Fig, 254.

The line $2x+5y=11$ has divided the plane of the paper into two half planes. The line itself is the boundary of two half planes. A half plane may or may not contain its boundary line. If it does, it is called a closed half plane, otherwise it is called an open half plane. Two half planes are the two sides of the line. The half plane above the line consists of points for which $2x+5y>11$. This half plane consists of points which are called points above the line and this half plane if it does not contain its boundry will be the graph of the inequality $2x+5y>11$. If it contains its boundary line, then it will be the graph of $2x+5y \geqslant 11$. All the points below the line also constitute a half plane. This half plane will be the graph of $2x+5y<11$. If it contains its boundary line it will be the graph of $2x+5y \leq 11$.

Let us consider the equation $y=3$. The graph of this equation divides the plane into two half planes. The half plane consisting of points which are above the line will be the graph of $y>3$, and the half plane consisting of points below this line will be the graph of $y<3$.

But if the line is not horizontal how do we describe the half planes above and below it ?

Consider the equation $y-2x=1$. It will be advisable to write it first in slope form *viz.*, $y=2x+1$.

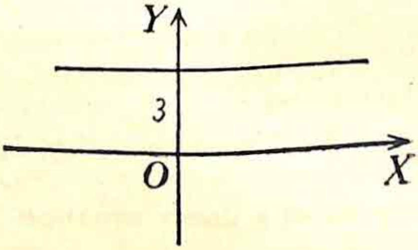


Fig. 255.

A point P belongs to a half plane above the line iff it is vertically above some point on the line.

Consider a point $(-1, 1)$, it is vertically above a point $(-1, -1)$ which is on the line, so this point belongs to a half plane containing points above the line.

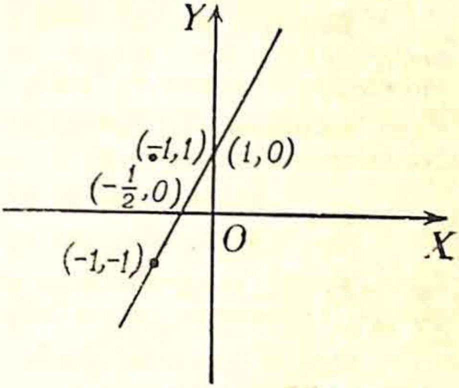


Fig. 256.

Alternatively. We know that $y=2x+1$. The x-co-ordinate of the point that we are considering is -1 so $y=-2+1$ or $y=-1$, so the y-co-ordinate of the point that we are considering is greater than -1.

Let us consider a point $(-1, -3)$.

The line is $y=2x+1$

when $x=-1$, then $y=-2+1$ or $y=-1$

but $-1>-3$. Hence $(-1, -3)$ belongs to the half plane that is below the line.

Points to the right and points to the left of a line.

Consider the equation $x=-5$. The graph of this line divides the plane of the paper into two half planes. Now this line is a vertical line so we cannot speak of points above and below the line. However we can speak of half plane to the right of the line and half plane to the left of the line. Every point is to the right of the line iff it is horizontally to the right of a point on the line.

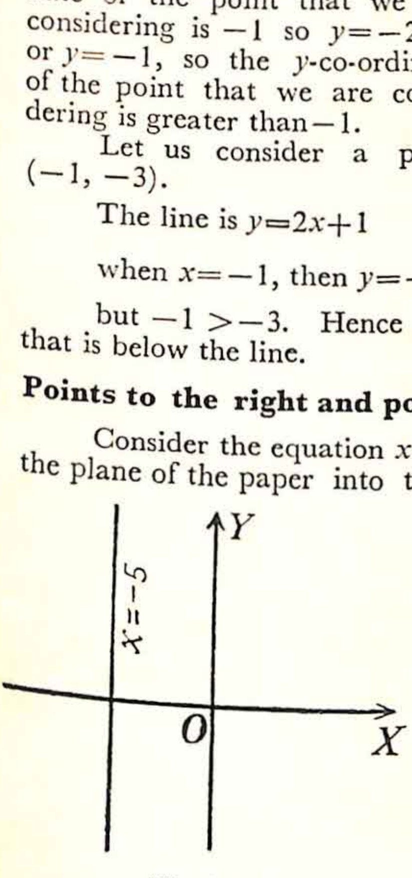


Fig. 257.

Consider a point $(-1, -3)$. Now a point $(-5, -3)$ will be on the line $x=-5$. $(-1, -3)$ is horizontally to the right of the point $(-5, -3)$ which is on the line. Therefore point $(-1, -3)$ belongs to a half plane to the right of the line. Clearly $(-6, -3)$ will belong to half plane to the left of the line. How to decide about a point when the boundary line is not vertical ? Consider the line $x+2y=2$.

Let us write it as $x=-2y+2$. Consider a point $(7, -3)$. Now $x=-2y+2$.

When $y=-3$, then $x=(-2)(-3)+2$ or $x=8$. Now the point $(8, -3)$ belongs to the line.

As $7<8$, the point $(7, -3)$ lies horizontally to the left of the point $(8, -3)$ which lies on the line. Therefore the point $(7, -3)$ lies horizontally to the left of the line. In other words $(7, -3)$ belongs to the half plane represented by the inequality $x<-2y+2$.

Example 30. Determine whether the point $(4, -2)$ is vertically above or below the line and also horizontally to the right or to the left of the line $x+2y=2$.

Solution. To determine whether $(4, -2)$ is vertically above or below the line we write

$$x+2y=2 \text{ as } 2y=-x+2 \text{ or } y=-\tfrac{1}{2}x+1.$$

Now when $x=4$; $y=-\frac{1}{2}\times 4+1$ or $y=-1$.

as $-2<-1$, the point lies in the half plane below the line and which is represented by the inequality $y<-\frac{1}{2}x+1$.

Now to determine whether $(4, -2)$ lies to the right or to the left of $x+2y=2$, we write the equation as $x=-2y+2$.

When $y=-2$ then $x=-2(-2)+2$ or $x=6$. As $4<6$, the point $(4, -2)$ lies horizontally to the left of $x+2y=2$.

Example 31. Graph the inequality $y-3x>1$.

Solution. First we graph the line $y-3x=1$. A(0, 1), B(1, 4), C(−1, −2) are the three points on the line. Now this line divides the plane into two half planes, one of these is a graph of $y-3x<1$ and the other is graph of $y-3x>1$. Now origin lies in one of the half planes. If we write $x=0$ and $y=0$ in $y-3x>1$ we get $0-0>1$ or $0>1$ which is not a true statement. Hence $(0, 0)$ lies in the half plane other than given by $y-3x>1$. Now consider a point $(0, 2)$ which lies in the half plane other than that contains the origin as if we write $x=0$ and $y=2$ in $y-3x>1$ we get $2-0>1$ or $2>1$ which is a true statement. Hence the half plane containing $(0, 2)$ is represented by $y-3x>1$. In fact, we should consider any one point in each of the half planes to decide as to which half plane represents the given inequality.

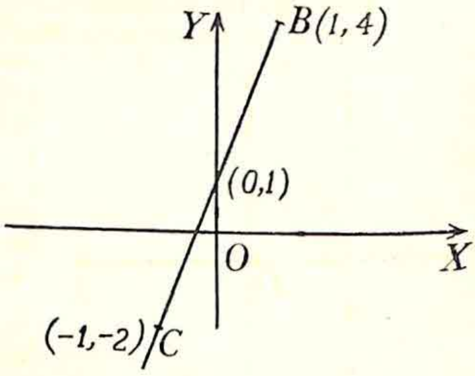


Fig. 258.

Problem Set

1. Determine whether each of the following points is above, or below, to the left or to the right of the line $y=2x+1$.

(*a*) $(2, 5)$, (*b*) $(0, 2)$, (*c*) $(-2, -3)$,
(*d*) $(-10, -20)$, (*e*) $(5, 12)$, (*f*) $(100, 210)$

2. Shade the half plane given by each of the inequality in two variables. (If boundary line does not belong to the graph then show it as a dotted line, otherwise show it by a thick line.)

(*a*) $y \geqslant -5$ (*b*) $y > -5$ (*c*) $x > -3$ (*d*) $x \geqslant -3$

(*e*) $y < 0$ (*f*) $x > 0$ (*g*) $x \geqslant 0$ (*h*) $y \leqslant 2x$

(*i*) $y < x$ (*j*) $y + 5x \geqslant 3$ (*k*) $y < -\frac{x}{4} + 1$.

3. Shade that portion of the plane which is simultaneously given by the following pairs of inequalities

(*a*) $x > y$ and $x > -y$ (*e*) $2x + y > 3$ and $y - 2x > 8$

(*b*) $x > y$ and $x < -y$ (*f*) $2x + y > 3$ and $y - 2x < 8$

(*c*) $x < y$ and $x > -y$ (*g*) $2x + y < 3$ and $y - 2x > 8$

(*d*) $x < y$ and $x < -y$ (*h*) $2x + y < 3$ and $y - 2x < 8$

Angle between two lines

If we are given a line p and two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ we define the slope of the line as

$$m = \frac{m\overline{QT}}{m\overline{PT}} = \frac{y_2 - y_1}{x_2 - x_1}$$

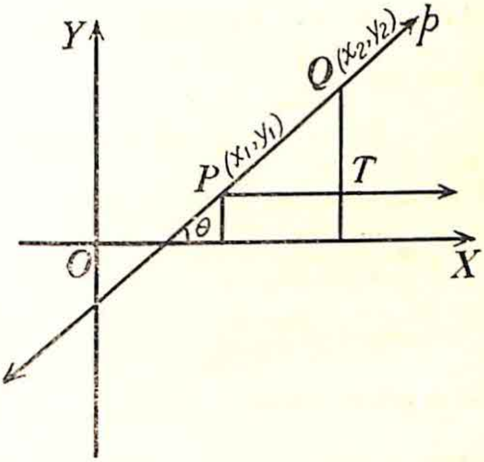


Fig. 259.

If the line p makes an angle θ with X-axis we have

$$m\angle QPT = \theta \text{ (corresponding angle).}$$

Also in trigonometry if $\angle QPT$ is an angle of a right $\triangle QPT$ than tan $\angle QPT$ is defined to be equal to

$$\frac{m\overline{QT}}{m\overline{PT}} = \frac{QT}{PT}.$$

Hence we can write that $m = \tan\theta$. It may be noted that $m\overline{QT}$ means the measure of the segment $\overline{QT}$ and m is not to be separated from $\overline{QT}$. We are also representing the slope of a line by m.

These two concepts should not be confused with each other. Now we are interested in finding the measure of the angle contained between the lines p and q. Let α be the measure of angle between p and q. If p makes an angle θ and q makes an angle β with the X-axis then we have already proved that $\beta=\alpha+\theta$ (which theorem ?), and

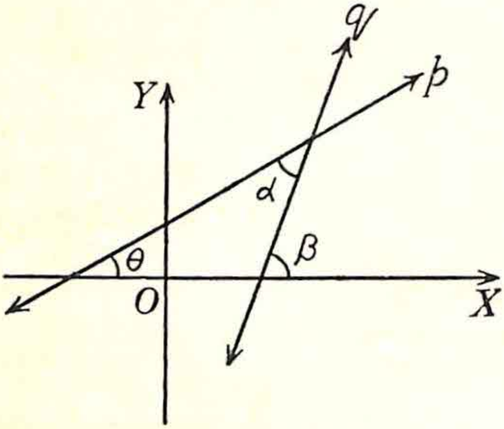


Fig. 260.

$$\beta=\alpha+\theta \leftrightarrow \alpha=\beta-\theta \leftrightarrow \tan\alpha=\tan(\beta-\theta)$$

$$\leftrightarrow \tan\alpha=\frac{\tan\beta-\tan\theta}{1+\tan\beta\tan\theta}$$

(a result proved in trigonometry)

If slope of q is m_1 and that of p is m_2 then

$$\tan\alpha=\frac{m_1-m_2}{1+m_1m_2}$$

Hence $$\alpha=\tan^{-1}\frac{m_1-m_2}{1+m_1m_2}$$

The angle α is acute or obtuse according as $\tan\alpha$ is +ve or —ve.

Example 32. Find the angle contained between the lines

$$3x+y=10 \text{ and } 3y+x=6$$

Solution. The slope of forms of the given lines are

$$y=-3x+10 \text{ and}$$

$$y=-\frac{1}{3}x+2$$

Hence $$m_1=-3 \text{ and}$$

$$m_2=-\frac{1}{3}$$

or
$$\tan \alpha=\frac{-3-\left(-\frac{1}{3}\right)}{1+(-3)\left(-\frac{1}{3}\right)}$$

$$=\frac{\frac{-8}{3}}{2}$$

$$=\frac{-4}{3}$$

or
$$\alpha=\tan^{-1}\left(\frac{-4}{3}\right)$$

Length of the perpendicular segment from a point to a line.

We have already discussed how to write the equation of a line in the perpendicular form and how to find the length of the perpendicular segment from the origin to the line. For example if the equation of a line p is $\sqrt{3}\,x+y=8$, then we can transform it into perpendicular form in the following manner :

$$\sqrt{3}\,x+y=8 \leftrightarrow \frac{\sqrt{3}\,x}{\sqrt{(\sqrt{3})^2+(1)^2}}+\frac{y}{\sqrt{(\sqrt{3})^2+(1)^2}}$$

$$=\frac{8}{\sqrt{(\sqrt{3})^2+(1)^2}} \leftrightarrow \frac{\sqrt{3}\,x}{2}+\frac{1}{2}\,y=4$$

then we know that the length of the perpendicular segment from the origin to the given line is 4 units. Sometimes we may be interested in finding the length of perpendicular segment from any point not lying on the line to a given line. Suppose we are interested in finding the length of perpendicular segment from P(2, 5) to this line.

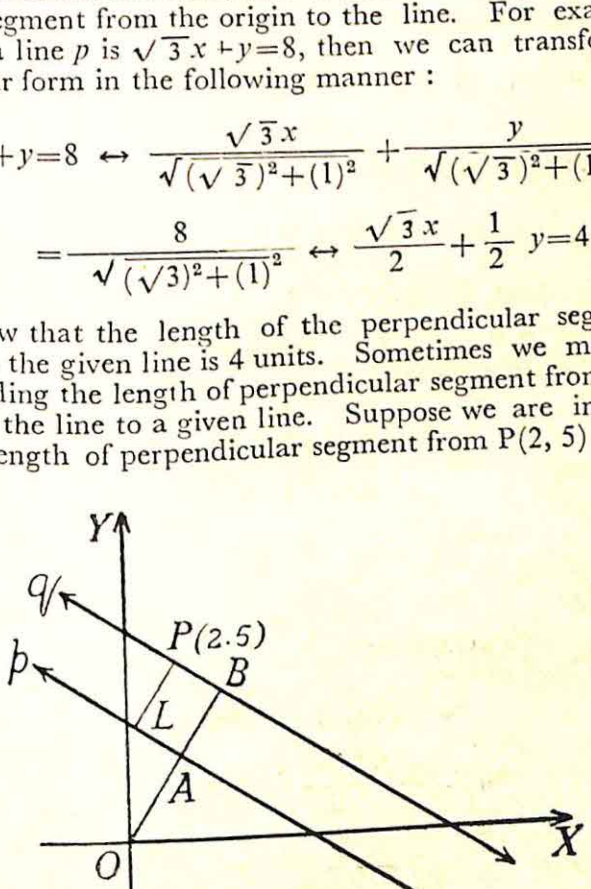


Fig. 261.

First we draw the graph of the line. Then we are interested in finding out whether P(2, 5) lies above or below this line. The given equation is

$$y=-\sqrt{3}x+8$$

when $x=2$,

then $y=-\sqrt{3}\times 2+8$

or $y=-1{\cdot}76\times 2+8$

$$=-3{\cdot}52+8$$

$$=4{\cdot}48$$

and as $5>4{\cdot}48$, therefore, the point (2, 5) lies above this line. Now we are interested in finding the length of the segment $\overline{PL}$. Let q be a line through P parallel to the given line. Clearly $\overline{OA}$ is the perpendicular segment on p from origin whose length we have already found. $\overline{OB}$ is the perpendicular segment from O to q. Now $m\overline{PL}=m(\overline{OB})-m(\overline{OA})$. To find $m\overline{OB}$ we observe that q is a line through (2, 5) whose slope is same as that of p.

Now the slope of $p=-\sqrt{3}$. Hence the slope of q is also $-\sqrt{3}$

Now the equation of q is

$$y-5=-\sqrt{3}\,(x-2)$$

$$\leftrightarrow \quad y+\sqrt{3}x=5+2\sqrt{3}$$

$$\leftrightarrow \quad \frac{y}{\sqrt{(1)^2+(\sqrt{3})^2}}+\frac{\sqrt{3}x}{\sqrt{(1)^2+(\sqrt{3})^2}}$$

$$=\frac{5+2\sqrt{3}}{\sqrt{(1)^2+(\sqrt{3})^2}}$$

$$\leftrightarrow \quad \frac{1}{2}y+\frac{\sqrt{3}}{2}x=\frac{5+2\sqrt{3}}{2}$$

Hence $m(\overline{OB})=\dfrac{5+2\sqrt{3}}{2}$

Hence $m(\overline{PL})=m(\overline{OB})-m(\overline{OA})$

or $m(\overline{PL})=\dfrac{5+2\sqrt{3}}{2}-4$

$$=\frac{2\sqrt{3}-3}{2}$$

Now let us consider the general case.

Let $ax+by+c=0$ be an equation of the line p and let $P(x_1, y_1)$ be the point from which the length of the perpendicular segment to p is to be determined.

Now to determine the slope of p we write it as

$$ax+by+c=0$$

$$\leftrightarrow \quad by=-ax-c \leftrightarrow y=-\frac{a}{b}x-\frac{c}{b}$$

Hence the slope of p is $-\frac{a}{b}$

Now the equation of q which contains P and is ∥ to p will be

$$y-y_1=-\frac{a}{b}(x-x_1)$$

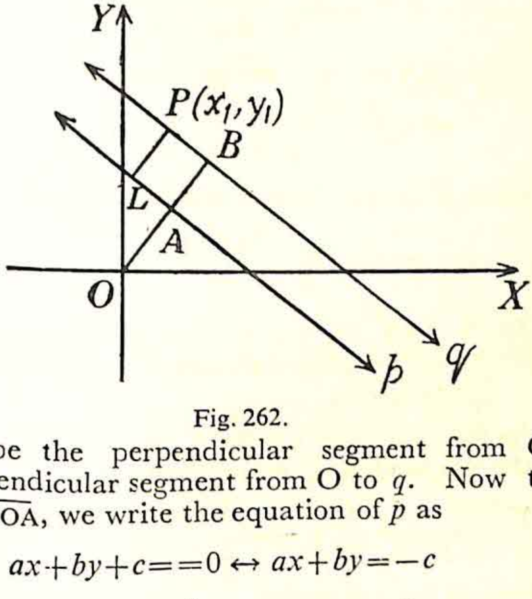


Fig. 262.

Let $\overline{OA}$ be the perpendicular segment from O to p and $\overline{OB}$ be the perpendicular segment from O to q. Now to determine the measure of $\overline{OA}$, we write the equation of p as

$$ax+by+c==0 \leftrightarrow ax+by=-c$$

$$\leftrightarrow \quad \frac{a}{\sqrt{a^2+b^2}}x+\frac{b}{\sqrt{a^2+b^2}}y=-\frac{c}{\sqrt{a^2+b^2}}$$

Hence $\quad m(\overline{OA})=-\frac{c}{\sqrt{a^2+b^2}}$

Now to find the measure of $\overline{OB}$, we write the equation of q as

$$y-y_1=-\frac{a}{b}(x-x_1)$$

$$\leftrightarrow \quad by-by_1=-ax+ax_1$$

$$\leftrightarrow \quad by+ax=by_1+ax_1$$

$$\leftrightarrow \quad ax+by=ax_1+by_1$$

$$\leftrightarrow \quad \frac{a}{\sqrt{a^2+b^2}}\,x+\frac{b}{\sqrt{a^2+b^2}}\,y=\frac{ax_1+by_1}{\sqrt{a^2+b^2}}.$$

Hence
$$m(\overline{\mathrm{OB}})=\frac{ax_1+by_1}{\sqrt{a^2+b^2}}$$
$$m(\overline{\mathrm{PL}})=m\overline{\mathrm{OB}}-m\overline{\mathrm{OA}}$$
$$=\frac{ax_1+by_1}{\sqrt{a^2+b^2}}-\left(\frac{-c}{\sqrt{a^2+b^2}}\right)$$
$$=\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$$

If P is below the line p, then $m(\overline{\mathrm{PL}})$ will be found to be negative or

$$m.\overline{\mathrm{PL}}=-\frac{(ax_1+by_1+c)}{\sqrt{a^2+b^2}}$$

and neglecting the negative sign we get that

$$m(\overline{\mathrm{PL}})=\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}.$$

Hence the measure of perpendicular segment from $\mathrm{P}(x_1, y_1)$ to

$ax+by+c=0$ equals $\left|\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

Now let us go back to our earlier problem of finding the length of the perpendicular segment from P(2, 5) to the line

$$\sqrt{3}\,x+y=8$$

On applying the formula that we have determined above we get the length of the perpendicular segment as :

$$\frac{\sqrt{3}\,(2)+5-8}{\sqrt{(\sqrt{3}\,)^2+(1)^2}}=\frac{2\sqrt{3}-3}{2}$$

which is the same as we got earlier.

Problem Set

(1) Find the distance from the line $x+2y-3=0$ to the point (7, 9)

(2) Show that the distance from line $ax+by-p=0$ to the point (x_1, y_1) is $\left|\dfrac{ax_1-by_1-p}{\sqrt{a^2+b^2}}\right|$

(3) Find the distance of the point (7, 9) from the line $3x-4y=5$

(4) Find the distance of the point $(-3, -4)$ to the line $-3x-4y=-5$.

CHAPTER XV

CIRCLES

In this chapter, we propose to discuss some of the important properties of sets of points which are called circles and with this we shall conclude our discussion about the Euclidean geometry. The sets of points that are called circles are not made up of planes, rays, lines and the like. These sets form curved figures which are the simplest among all the curved figures. Euclid postulated the existence of a circle with a given centre and given radius. One of his postulates was 'To describe a circle with any centre and distance'. In other words he meant that a circle with any centre and any radius exists. We, however, will define a circle as a set of points, having the following properties :—(1) The points are coplanar (2) Each member of the set will be at the same distance from a fixed point.

Definition XV (1). (*A circle*). The set of all those points that lie in a given plane E and whose distance from a given point P (P belonging to E) is a given positive number, r, is called a circle (denoted by ⊙) with centre P and radius r.

We can define a circle in terms of locus also. It may, however, be clearly understood that circle is a set of points whose distance from a given point P is r but P does not belong to the circle. All those points of E whose distance from P is less than r form a distinct set which is called the interior of the circle. Of course P itself belongs to the interior of the circle. So any point belonging to E belongs to either (*i*) interior of the circle (*ii*) circle or exterior of the circle.

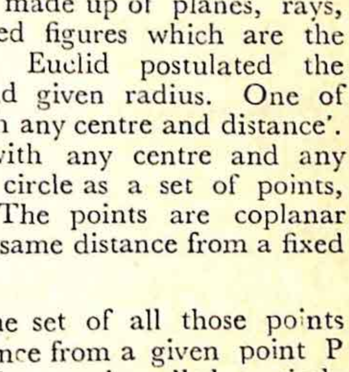


Fig. 263.

Also we define the following terms.

Definition XV (2). (*Chord*). A line segment whose end points belong to the circle, is called a *chord*. Clearly the chord lies in the plane of the circle as it contains two points of the plane.

Definition XV (3). (*Diameter*). A chord which contains centre of the circle as its element, is called the diameter of the circle.

Definition XV (4). (*Secant*). A line which contains a chord of a circle is called a secant of the circle.

Clearly, a secant contains two points of the circle.

Definition XV (5). (*Tangent*). A line which lies in the plane of the circle and intersects the circle in one and only one point, is called the tangent to the circle. The point of intersection of a tangent and the circle is called the *point of contact* of the circle or *point of tangency*. Two circles which are tangent to the same line at the same point are called tangent to each other. If tangent circles are coplanar then those are internally or externally tangent according as their centres lie on the same side or opposite side of the common tangent line.

Definition XV (6) (*Concentric circles*). Two circles which have the same point as centre are called concentric circles.

Definition XV (7). (*Congruent circles*). Two circles which have the same radius are said to be congruent circles.

Definition XV (8). (*Radial segment*). The line segment determined by the centre and any point belonging to the circle is called the radial segment.

It may be clearly understood that radius is a number r which is the measure of radial segment. Of course in our discussion, we will use the term radius for radial segment when there is no possibility of any confusion between the radial segment and its measure.

If we are given three non-collinear points (of course any three points are always coplanar), we can uniquely determine a circle that will contain these points. We can also determine the centre of the circle. In this regard, we prove the following theorem :—

Theorem XV (1). There is one and only one circle that contains three given non-collinear points.

Given : Three non-collinear points A, B and C.

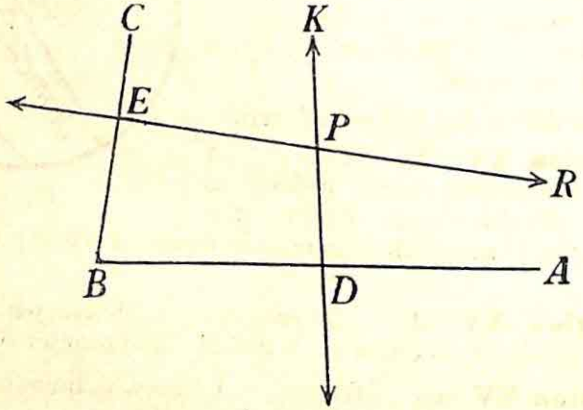


Fig. 264.

To prove that : (*i*) There exists a circle containing A, B and C.

(*ii*) There exists only one circle containing A, B and C.

Proof.

Statement	*Reason*
(1) A, B and B, C determine distinct line segments $\overline{AB}$ and $\overline{BC}$	(1) A, B, C are non-collinear
(2) $\overleftrightarrow{KD}$ the right bisector of $\overline{AB}$ and $\overleftrightarrow{RE}$ the right bisector of $\overline{BC}$ have one and only one point (P) in common	(2) $\overleftrightarrow{KD} \nparallel \overleftrightarrow{RE}$ as $\overline{AB}$ and $\overline{BC}$ have a point B in common. Also $\overleftrightarrow{KD} \neq \overleftrightarrow{RE}$ as neither $\overline{AB} = \overline{BC}$ nor $\overline{AB} \parallel \overline{BC}$
(3) PA=PB=PC	(3) Theorem on locus (which ?)
(4) The distance of each of A, B, C from P is same and also A, B, C are coplanar.	(4) Statement 3. Given
(5) There exists a circle containing A, B and C and whose centre is P	(5) Statement 4. Definition of a circle
(6) $\overleftrightarrow{KD}$ and $\overleftrightarrow{ER}$ have only one point in common	(6) Statement 2
(7) There is only one point P whose distance from each of A, B, C is same	(7) Statements 4 and 6
(8) There is only one circle that contains A, B and C.	(8) Statement 7

Theorem XV (2). If two circles C_1 and C_2 have three points in common then $C_1 = C_2$.

Proof. Its proof immediately follows from the theorem XV (1).

Theorem XV (3). Two distinct circles C_1 and C_2 cannot have more than two points in common.

Proof. Assume that C_1 and C_2 have three points in common then $C_1 = C_2$ [Theorem XV (2)] but this contradicts the hypothesis that $C_1 \neq C_2$. Hence the assumption that C_1 and C_2 have three points in common has to be abandoned.

Theorem XV (4). If P is a point in the interior of the circle whose distance from each of the points A, B, C (lying on the same circle) is the same, then P is the centre of the circle.

Proof. Its proof follows from theorem XV (1).

An arbitrary line lying in the plane of a circle may entirely lie in the exterior of the circle (in other words it may not intersect the circle at all) or it may intersect the circle in one point or it may intersect it in two points. Now what are the conditions under which a line will adopt any one of these three courses ? For this, we prove the following theorem :—

Theorem XV (5). Given a circle C with centre P and radius r. Also given an arbitrary line l in the plane E of the circle. F is the foot of the perpendicular from P to l and $PF=p$ then :—

(1) l does not intersect the circle if $p>r$.

(2) l intersects the circle only in one point if $p=r$.

(3) l intersects the circle in two distinct points which are equidistant from F if $p<r$.

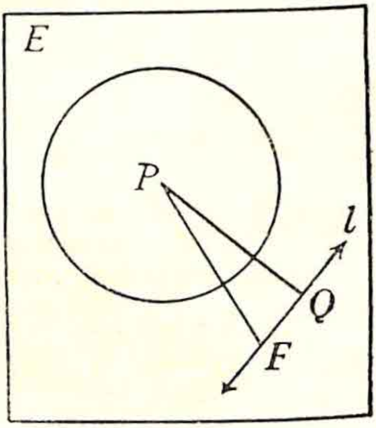


Fig. 265.

Given : A circle C with centre P and radius r. A line l lying in the plane E of the circle. F is the foot of perpendicular from P to l and $PF=p$.

To prove that : (*i*) l does not intersect the circle if $p>r$.

(*ii*) l intersects the circle in one and only one point if $p=r$.

(*iii*) l intersects the circle in two distinct points which are equidistant from F if $p<r$.

Proof. First Part. If $p>r$ then F lies in the exterior of the circle as the distance of any point in the interior of the circle or on the circle from its centre will be either less than or equal to r.

Statement	*Reason*
(1) $\overline{PF}$ is the shortest segment joining P to l	(1) Why ?
(2) $PQ>PF$ where $Q\in l$ and $Q\neq F$	(2) Statement 1
(3) $PQ>r$	(3) $PQ>PF$ and $PF>r$
(4) Q lies in the exterior of the circle	(4) Statement 3

(5) Every point of l lies in the exterior of the circle	(5) As F lies in the exterior of the circle and Q, any arbitrary point of l, also lies in the exterior of the circle
(6) l does not intersect the circle C.	(6) Statement 5.

Secod Part. If $p=r$.

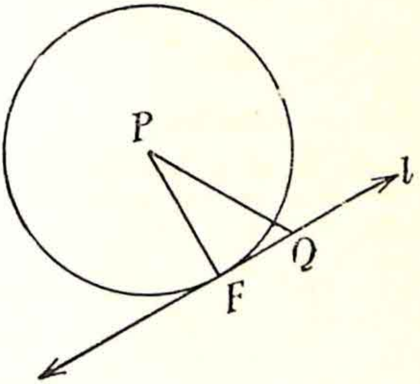


Fig. 266.

Statement	*Reason*
(1) F is a point on the circle	(1) $p=r$
(2) l intersects the circle in at-least one point	(2) Statement 1
(3) $\overline{PF}$ is the shortest segment joining P to l	(3) Why ?
(4) $PQ>PF$ where $Q \in l$ and $Q \neq F$	(4) Statement 3
(5) $PQ>r$	(5) $PQ>PF$ and $PF=r$
(6) Q does not belong to circle	(6) Statement 5
(7) l intersects the circle in only one point F	(7) Statements 2 and 6

Third Part. If $p<r$. F lies in the interior of the circle (why ?) If a point Q is both on the line and the circle, then as $\triangle PFQ$ is a right triangle with $m\angle F=90$ then :

$$(PQ)^2=(PF)^2+(FQ)^2$$

$$\leftrightarrow \quad r^2=p^2+(FQ)^2$$

$$\leftrightarrow (FQ)^2=r^2-p^2 \leftrightarrow FQ=\sqrt{r^2-p^2}$$

It may be noted that r^2-p^2 is a positive number as $p<r$ (why ?) and since every positive number has a positive square root

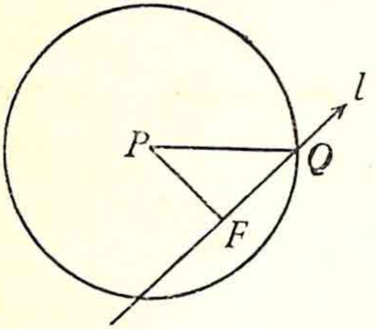


Fig. 267.

(Euclidean completeness Postulate ; chapter II), FQ is a positive number. By point plotting theorem, there are exactly two points Q_1 and Q_2 on the opposite side of F such that

$$FQ_1=FQ_2=\sqrt{r^2-p^2}. \text{ Moreover for these points}$$
$$PQ_1=PQ_2=\sqrt{p^2+(\sqrt{r^2-p^2})^2}=r \text{ (How ?)}$$

Hence Q_1 and Q_2 are on the circle. A situation in which this reasoning will not apply will be when l contains P but in this case P=F and $PQ_1=PQ_2=r$ and there are two points Q_1 and Q_2 on the circle satisfying the condition and lying on the given line. Of course in this case, the line l becomes a diameter.

Theorem XV (6). Any perpendicular from the centre of a circle C to a chord, bisects the chord.

Given : A circle with centre P and $\overline{AB}$ is any chord of the circle. $\overline{PF} \perp \overline{AB}$ where F is the foot of perpendicular from P to $\overline{AB}$.

To prove that : FA=FB.

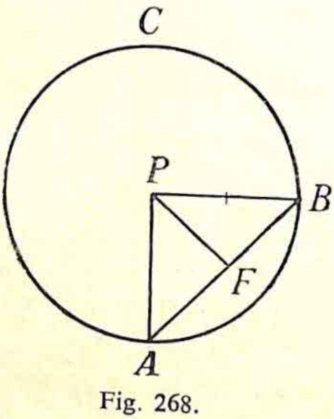


Fig. 268.

Proof.

Statement	*Reason*
(1) $\overline{PA} \cong \overline{PB}$	(1) Radii of the same circle
(2) $\overline{PF} \cong \overline{PF}$	(2) Any segment is congruent to itself
(3) 1 – 1 correspondence PFA≅PFB is a congruence and $\triangle$PFA≅$\triangle$PFB	(3) Hyp. leg. Theorem
(4) $\overline{FA} \cong \overline{FB}$.	(4) Corresponding parts of congruent triangles.

[This theorem is a direct corollary to the third part of theorem XV (5) and could have been proved as such].

We can prove the converse of this theorem also as if F is the mid point of $\overline{AB}$ then we can prove that $\overline{PF} \perp \overline{AB}$ since 1 – 1 correspondence PFA↔PFB will be a congruence and $\triangle$PFA≅$\triangle$PFB as $\overline{FA} \cong \overline{FB}$ (S.S.S. theorem). It would then imply that $\angle$PFA≅$\angle$PFB and since $\angle$PFA, $\angle$PFB form a linear pair, the measure of each will be 90.

We can state the converse as :

Theorem XV (7). The segment joining the centre of a circle C to the mid point of a chord is perpendicular to the chord.

Theorem XV (8). Every line tangent to a circle C is perpendicular to the radius drawn to the point of contact and conversely any line in the plane of the circle, perpendicular to a radius of the circle at its outer end is tangent to the circle.

Given : A circle with centre P. A line *l* which is tangent to the circle at a point F (As the line is tangent to the circle, it intersects it in one and only one point).

To prove that : $\overline{PF} \perp l$.

Proof. The proof of this theorem can be directly deduced from the second part of theorem XV (5). Alternatively we can proceed as follows :

Let us assume the contrary and state that PF is not perpendicular to *l*. Let PF′ be the perpendicular segment from P to *l* where $F' \in l$ but $F' \neq F$, Then :

Statement	*Reason*
(1) $PF' < PF$	(1) Why ?
(2) $PF = r$	(2) F belongs to the circle
(3) $PF' < r$	(3) Statements 1 and 2

(4) F′ is in the interior of the circle	(4) Statement 3
(5) l intersects the circle in two points	(5) Theorem XV (5). third part.

Statement 5 contradicts the hypothesis that l is a tangent to the circle. Hence the assumption that PF is not perpendicular to l has to be abandoned.

Proof of the converse.

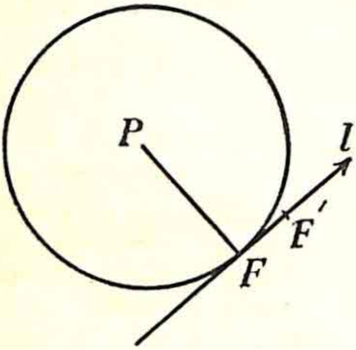


Fig. 269.

Given : A line l, lying in the plane of the circle C with centre P and perpendicular to radius $\overline{PF}$ at its outer end F.

To prove that : l is a tangent to the circle or l intersects the circle in one and only point.

Proof. Its proof can be deduced from the second part of theorem XV (5). Alternatively we can proceed as follows :

(1) l intersects the circle in atleast one point as the point F belongs to the circle as well as to the line.

(2) Let us assume that the line intersects the circle in another point F′ (F≠F′) Then :

Statement	*Reason*
(1) $\overline{PF} \cong PF'$	(1) F and F′ both belong to the circle
(2) PF′>PF	(2) As PF ⊥ l.

Now the statements (1) and (2) are contradictory to each other. Hence the assumption that l intersects the circle in two points has to be abandoned.

Theorem XV (9). The perpendicular bisector of a chord of a circle and lying in its plane, passes through the centre of the circle

Given : A circle C with centre P. $\overline{AB}$ is any chord of the circle. F is mid point of $\overline{AB}$, A line *l* is perpendicular bisector of $\overline{AB}$ and lies in the plane of the circle.

To prove that : *l* passes through P.

Proof. Its proof can be directly deduced from the proof of theorem XV (5) (First part). Alternatively assume that *l* does not pass through P (Refer Fig. 268). Now $\overline{PF}$ is perpendicular to $\overleftrightarrow{AB}$ (why ?) Hence we have $l \neq \overleftrightarrow{PF}$ and *l* and $\overleftrightarrow{PF}$ both are perpendicular to $\overleftrightarrow{AB}$ at the same point F which contradicts a theorem which we have proved earlier (which ?) Hence the assumption that *l* does not pass through P has to be abandoned.

Theorem XV (**10**). If two circles are tangent to each other, their point of contact lies on the line through their centres.

(By definition, two circles are tangent to each other, if they are tangent to the same line at the same point. If tangent circles are coplanar then these are internally or externally tangent according as their centres lie on the same side or opposite side of the common tangent line).

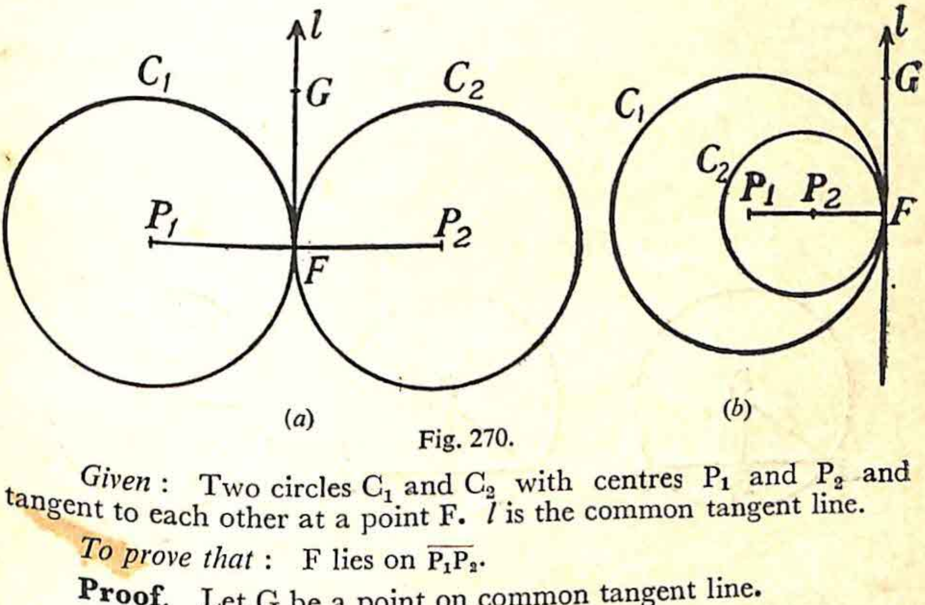


Fig. 270.

Given : Two circles C_1 and C_2 with centres P_1 and P_2 and tangent to each other at a point F. *l* is the common tangent line.

To prove that : F lies on $\overline{P_1P_2}$.

Proof. Let G be a point on common tangent line.

Case First. When the circles are tangent to each other externally [Fig. 270 (*a*)].

Statement	*Reason*
(1) $m\angle GFP_1=90$ and $m\angle GFP_2$ $=90$	(1) Why ?

(2) $m\angle GFP_1 + m\angle GFP_3 = 180$	(2) Why ?
(3) $\angle GFP_1$, $\angle GFP_2$ form a linear pair	(3) Why ?
(4) $\overrightarrow{FP_1}$ and $\overrightarrow{FP_2}$ are opposite rays	(4) Statement 3.
(5) F lies on $\overleftrightarrow{P_1P_2}$	(5) Statement 4.

Case Second. When the circles are tangent to each other internally,

Assume that $\overleftrightarrow{FP_1} \neq \overleftrightarrow{FP_2}$. Now $\angle GFP_1 = 90$ (Why ?) $\angle GFP_2 = 90$ (Why ?) $\angle GFP_1 \cong \angle GFP_2$ (why ?) $\overleftrightarrow{GF} = \overleftrightarrow{GF}$. Hence $\overleftrightarrow{FP_1} = \overleftrightarrow{FP_2}$ (Protractor Postulate). Hence the assumption that $\overleftrightarrow{FP_1} \neq \overleftrightarrow{FP_2}$ has to be abandoned. F lies on $\overleftrightarrow{P_1P_2}$.

Theorem XV (11). In the same circle (or in congruent circles), chords equidistant from the centre are congruent and conversely.

Given : A circle C with centre P. Chords $\overline{AB}$ and $\overline{CD}$. $\overline{PF_1} \perp \overline{AB}$ and $\overline{PF_2} \perp \overline{CD}$ where F_1 is the foot of perpendicular from P to $\overline{AB}$ and F_2 is the foot of perpendicular from P to $\overline{CD}$. Also $\overline{PF_1} \cong \overline{PF_2}$ (Distance of a chord from the centre implies the measure of perpendicular segment from the centre to the chord).

To prove that : $\overline{AB} \cong \overline{CD}$

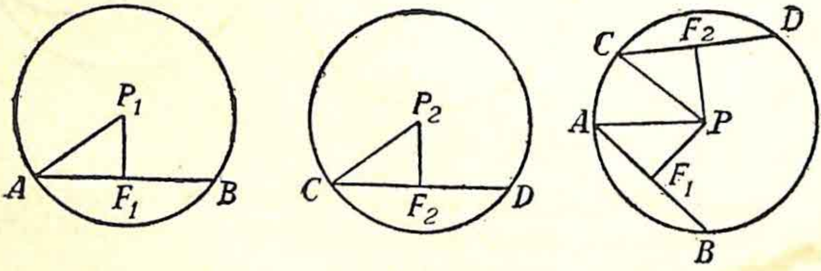


Fig. 271.

Proof.

Statement	*Reason*
(1) $\overline{PA} \cong \overline{PC}$ and $\overline{PF_1} \cong \overline{PF_2}$	(1) Why ?
(2) $m\angle PF_1A = m\angle PF_2C = 90$	(2) Why ?

(3) 1—1 correspondence $PF_1A \leftrightarrow PF_2C$ is a congruence and $\triangle PF_1A \cong \triangle PF_2C$	(3) Hypotenuse-leg theorem
(4) $\overline{AF_1} \cong \overline{CF_2}$	(4) Corresponding parts of congruent triangles
(5) $AB = 2AF_1$ and $CD = 2CF_2$	(5) Theorem XV (6)
(6) $\overline{AB} \cong \overline{CD}$	(6) Statements 4 and 5.

On similar lines, we can prove that in congruent circles, chords equidistant from the centre are congruent.

Converse to the theorem.

Statement. In the same circle (or in congruent circles) congurent chords are equidistant from the centre.

Given : A circle C with centre P. $\overline{AB}$, $\overline{CD}$ are chords of C such that $\overline{AB} \cong \overline{CD}$. $\overline{PF_1}$ is $\perp$ $\overline{AB}$ and $\overline{PF_2}$ $\perp$ $\overline{CD}$. $F_1 \in \overline{AB}$ and $F_2 \in \overline{CD}$.

To prove that : $\overline{PF_1} \cong \overline{PF_2}$.

Proof.

Statement	*Reason*
(1) $\overline{AB} \cong \overline{CD}$	(1) Given
(2) $\overline{AF_1} \cong \overline{CF_2}$	(2) F_1 is mid point of $\overline{AB}$ and F_2 is mid point of $\overline{CD}$ [theorem XV (6)]. Also statement 1.
(3) $\overline{PA} \cong \overline{PC}$	(3) Why ?
(4) $m\angle PF_1A = m\angle PF_2C = 90$	(4) Why ?
(5) 1—1 correspondence $PF_1A \leftrightarrow PF_2C$ is a congruence and $\triangle PF_1A \cong \triangle PF_2C$	(5) Hypotenuse-leg theorem
(6) $\overline{PF_1} \cong \overline{PF_2}$	(6) Corresponding parts of congruent triangles.

Some of the very important properties of the circles involve the notion of circular arcs, their measures and angles related to these. In order to discuss, we define the following :

Definition. XV (9). (*Central angle*). A central angle of a circle is an angle whose vertex is the centre of the circle.

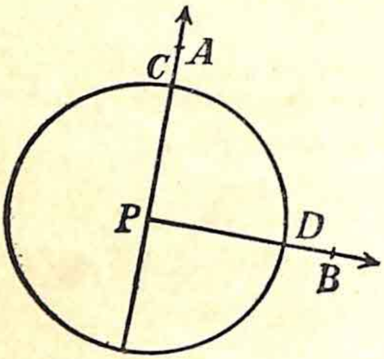


Fig. 272.

In this figure, P is the centre of the circle and $\overrightarrow{PA}$, $\overrightarrow{PB}$ are two rays containing radii $\overline{PC}$ and $\overline{PD}$. $\angle APB$ is a central angle of the circle.

Definition. XV (10). (*Minor arc*). If C, D are any two points of a circle such that C, D are not the end points of a diameter, then the union of C, D and all points of the circle which lie in the interior of $\angle CPD$ is called a minor arc $\overset{\frown}{CD}$ of the circle.

Definition. XV (11). (*Major arc*). If C, D are any two points of a circle such that C, D are not the end points of a diameter then the union of C, D and all points of the circle which lie in the exterior of $\angle CPD$ is called a major arc $\overset{\frown}{CD}$ of the circle.

Definition. XV (12). (*Semicircle*). If $\overline{EF}$ is a diameter of a circle which divides the plane of the circle into two half planes H_1 and H_2, then the union of A, B and all points of the circle that lie in H_1 will determine a semi-circle. Also the union of A, B and all points of circle lying in H_2 will determine another semi-circle.

Minor arcs, semicircles and major arcs are together referred to as arcs of the circle. Of course denoting an arc determined by points C, D on a circle by $\overset{\frown}{CD}$ will lead to ambiguity because C, D will determine two arcs—either a major arc and a minor arc or two semicircles. Whenever there is a possibility of this ambiguity, we take an interior point (say x) on the arc (x will of course be any point of the arc excepting the end points C or D) and then denote the arc by $\overset{\frown}{CXD}$ or $\overset{\frown}{DXC}$. If there is no chance of any doubt then we simply denote the arc by $\overset{\frown}{CD}$ or $\overset{\frown}{DC}$. If $\overset{\frown}{CXD}$ is a minor arc of the circle and if y is a point of the circle that does not belong to $\overset{\frown}{CXD}$ then $\overset{\frown}{CYD}$ is called the major arc corresponding to the minor arc $\overset{\frown}{CXD}$ and $\overset{\frown}{CXD}$ is called the minor arc corresponding to major arc $\overset{\frown}{CYD}$. If $\overset{\frown}{EF}$ is a diameter then $\overset{\frown}{EXF}$ and $\overset{\frown}{EYF}$ are semicircles.

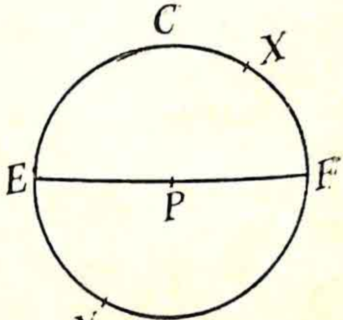


Fig. 273.

We are also interested in knowing the degree measure of an arc $\overset{\frown}{AXB}$ which we denote by $m\,\overset{\frown}{AXB}$. We define it as follows :

Definition XV (13). *(Degree measure of an arc).* The degree measure of $\overset{\frown}{AXB}$ or $m\,\overset{\frown}{AXB}$ is defined as :—

(1) Measure of corresponding central angle $\angle PAB$ if $\overset{\frown}{AXB}$ is a minor arc.

(2) 180 if $\overset{\frown}{AXB}$ is a semicircle.

(3) 360 − the degree measure of corresponding minor arc if $\overset{\frown}{AXB}$ is a major arc.

Even we can call a circle as an arc without any end points and assign degree measure of 360 to it.

Definition XV. (14). (*Congruent arcs*), In the same circle or in congruent circles, two arcs are congruent, if these have the same degree measure.

Of course, the intutive idea underlying the congruent arcs is same as underlies congruent segments, angles, triangles, polygons or circles. Now we discuss some theorems pertaining to arcs.

Theorem XV. (12). In the same circle (or in congruent circles) if two chords are congruent so are the corresponding minor arcs and conversely.

Given : A circle C with centre P. Chord $\overline{AB}$ of a circle is congruent to another chord $\overline{A'B'}$ of the circle. Chords $\overline{AB}$ and $\overline{A'B'}$ determine corresponding minor arcs $\overset{\frown}{AB}$ and $\overset{\frown}{A'B'}$.

To prove that : $m\,\overset{\frown}{AB} = m\,\overset{\frown}{A'B'}$ or in other words $\angle ABP \cong \angle A'P'B'$.

Proof :

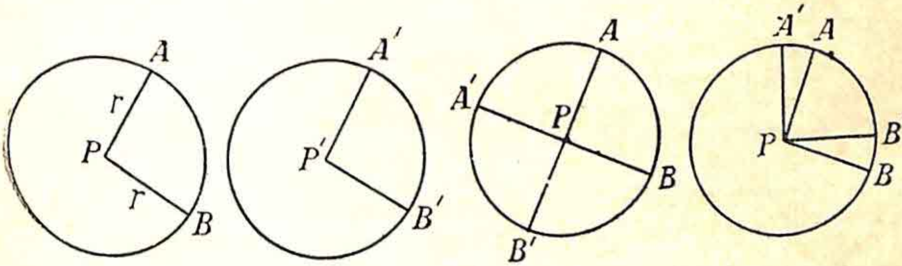


Fig. 274.

Statement	Reason
(1) $\overline{AB} \cong \overline{A'B'}$	(1) given
(2) $\overline{PA} \cong \overline{P'A'}$	(2) given

(3) $\overline{PB} \cong \overline{P'B'}$	(3) why ?
(4) 1—1 correspondence $PAB \leftrightarrow P'A'B'$ is a congruence and $\triangle PAB \cong \triangle P'A'B'$.	(4) S.S.S. Theorem
(5) $\angle APB \cong \angle A'P'B'$	(5) corresponding parts of congruent triangles
(6) $m\,\widehat{AB} = m\,\overset{\frown}{A'B'}$	(6) statement 5.

Converse to the theorem : In the same circle or in congruent circles, if two arcs are congruent then so are the corresponding chords.

Given : A circle C with centre P. Also chords $\overline{AB}$ and $\overline{A'B'}$ of the circle such that the degree measure of minor arc $\widehat{AB}$ equals the degree measure of minor arc $\overset{\frown}{A'B'}$.

To prove that : $\overline{AB} \cong \overline{A'B'}$

Proof :

Statement	*Reason*
(1) $\angle APB \cong \angle A'PB'$	(1) given that $m\,\widehat{AB} = m\,\overset{\frown}{A'B'}$
(2) 1—1 correspondence $APB \leftrightarrow A'PB'$ is a congruence and $\triangle APB \cong \triangle A'PB'$	(2) S.A.S. Postulate
(3) $\overline{AB} \cong \overline{A'B'}$	(3) Corresponding parts of congruent triangles.

By proceeding on similar lines, we can prove the case when congruent arcs are taken on congruent circles. We can apply this theorem to prove that 'In the same circle (or in congruent circles) if two chords are congruent, so are the corresponding major arcs and conversely.'

Should we expect that if an arc is a union of two arcs then the degree measure of the arc be equal to the sum of the degree measures of constituent arcs ? Let us discuss the following theorem in this regard.

Theorem XV (13). If $\overset{\frown}{AXB}$ and $\overset{\frown}{BYD}$ are the arcs of the same circle, having only the end point B in common, then $m\overset{\frown}{AXB} + m\overset{\frown}{BYD} = m\overset{\frown}{ABD}$. (In fact we always have $m\overset{\frown}{ABD} = m\widehat{AB} + m\widehat{BD}$).

Given : $\overset{\frown}{AXB}$ and $\overset{\frown}{BYD}$ of a circle C with centre P, having end point B in common.

To prove that : $m\overset{\frown}{AXB}+m\overset{\frown}{BYD}=m\overset{\frown}{ABD}$.

Proof. We have to discuss the following cases.

(1) When $\overset{\frown}{AXB}$, $\overset{\frown}{BYD}$ and $\overset{\frown}{ABD}$ are all minor arcs.

(2) When $\overset{\frown}{AXB}$ and $\overset{\frown}{BYD}$ are minor arcs but $\overset{\frown}{ABD}$ is a semicircle.

(3) When $\overset{\frown}{AXB}$ and $\overset{\frown}{BYD}$ are minor arcs but $\overset{\frown}{ABD}$ is a major arc.

(4) When one of $\overset{\frown}{AXB}$ or $\overset{\frown}{BYD}$ is a semicircle while the other is a minor arc.

(5) When one of $\overset{\frown}{AXB}$ or $\overset{\frown}{BYD}$ is a major arc.

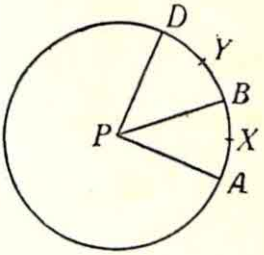


Fig. 275.

First Case. When each of $\overset{\frown}{AXB}$, $\overset{\frown}{BYD}$ and $\overset{\frown}{ABD}$ is a minor arc.

Statement	*Reason*
(1) B lies in the interior of $\angle APD$	(1) $\overset{\frown}{ABD}$ is a minor arc and B is interior point of $\overset{\frown}{AD}$
(2) $\overrightarrow{PB}$ is between $\overrightarrow{PA}$ and $\overrightarrow{PD}$	(2) Statement 1. Definition of interior of an angle.
(3) $m\angle APB+m\angle BPD=m\angle ABD$	(3) Statement 2. Angle addition Postulate
(4) $m\angle APB=m\overset{\frown}{AXB}$ $m\angle BPD=m\overset{\frown}{BYD}$ $m\angle APD=m\overset{\frown}{ABD}$	(4) As each of $\overset{\frown}{AXB}$, $\overset{\frown}{BYD}$ and $\overset{\frown}{ABD}$ is a minor arc
(5) $m\overset{\frown}{AXB}+m\overset{\frown}{BYD}=m\overset{\frown}{ABD}$	(5) Statements 3 and 4.

Second Case. When $\overset{\frown}{AXB}$ and $\overset{\frown}{BYD}$ are minor arcs but $\overset{\frown}{ABD}$ is a semicircle.

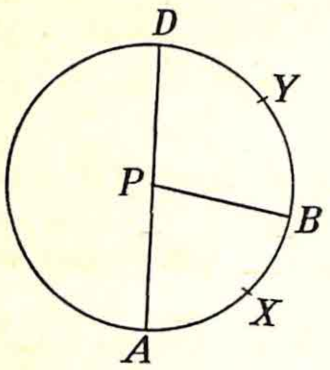


Fig. 276.

Statement	*Reason*
(1) $\overline{AD}$ is a diameter of the circle	(1) $\overset{\frown}{ABD}$ is a semicircle.
(2) $\angle APB$ and $\angle BPD$ form a linear pair	(2) $\overrightarrow{PA}$ and $\overrightarrow{PD}$ are opposite rays. Statement 1
(3) $m\angle APB + m\angle BPD = 180$	(3) Statement 2
(4) $m\angle APB = m\overset{\frown}{AXB}$ $m\angle BPD = m\overset{\frown}{BYD}$	(4) $\overset{\frown}{AXB}$ and $\overset{\frown}{BYD}$ are minor arcs
(5) $m\overset{\frown}{ABD} = 180$	(5) $\overset{\frown}{ABD}$ is a semicircle
(6) $m\overset{\frown}{AXB} + m\overset{\frown}{BYD} = m\overset{\frown}{ABD}$	(6) Statements 3, 4 and 5

Third Case. When $\overset{\frown}{AXB}$, $\overset{\frown}{BYD}$ are minor arcs but $\overset{\frown}{ABD}$ is a major arc.

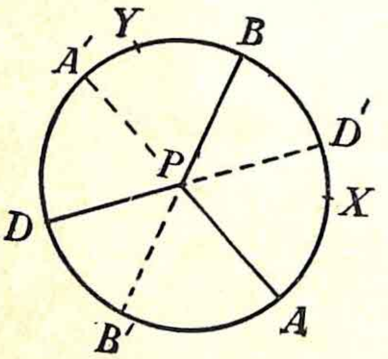


Fig. 277

Let B′ be the other end point of the diameter of the circle whose one end point is B. Let A′ and D′ be the end points of the diameters of the circle whose other end points are respectively A and D.

Statement	*Reason*
(1) B and D are on the opposite sides of $\overline{AA'}$	(1) $\overset{\frown}{AXB}$ is a minor arc while $\overset{\frown}{ABD}$ is a major arc
(2) B and B′ are on opposite sides of $\overline{AA'}$	(2) Since $\overline{AA'}$ and BB′ are distinct diametes
(3) B′, D lie on the same side of AA′ or $\overleftrightarrow{AP}$ or B′ lies on D-side of $\overleftrightarrow{AP}$	(3) Statements 1 and 2
(4) B and A lie on the oppsite sides of $\overline{DD'}$	(4) $\overset{\frown}{BYD}$ is a minor arc while $\overset{\frown}{ABD}$ is a major arc
(5) B′, A lie on the same side of $\overline{DD'}$ or $\overleftrightarrow{DP}$ or B′ lies on A-side of $\overleftrightarrow{DP}$	(5) As, B, B′ lie on opposite sides of DD′ Also statement 4
(6) B′ lies in the interior of $\angle APD$	(6) Statements 3 and 5
(7) $m\angle APD = m\angle APB' + \angle DPB'$	(7) Statement 6
(8) $m\angle APB' = 180 - m\angle APB$ $= 180 - m\overset{\frown}{AXB}$	(8) Why ?
(9) $m\angle DPB' = 180 - m\angle DPB$ $= 180 - m\overset{\frown}{BYD}$	(9) Why ?
(10) $m\overset{\frown}{AB'D} = 360 - m\overset{\frown}{ABD}$	(10) Since $\overset{\frown}{ABD}$ is a major arc
(11) $m\overset{\frown}{AB'D} = 180 - m\overset{\frown}{AXB} + 180 - m\overset{\frown}{BYD}$	(11) Statements 8, 9 and 7 and since $m\,\overset{\frown}{AB'D} = m\angle APD$

(12) $360 - m\overset{\frown}{ABD} = 180 - m\overset{\frown}{AXB} + 180 - m\overset{\frown}{BYD}$	(12) Statements 8 and 13
(13) $m\overset{\frown}{ABD} = m\overset{\frown}{AXB} + m\overset{\frown}{BYD}$	

Fourth Case. When either of $\overset{\frown}{BYD}$ or $\overset{\frown}{AXB}$ is a semicircle and the other is a minor arc.

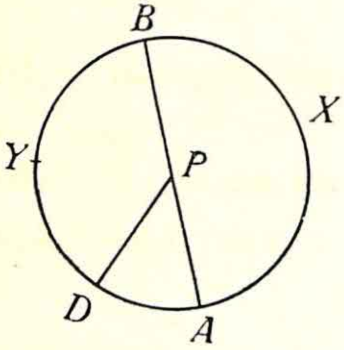


Fig.278

In this case $\overset{\frown}{ABD}$ will be a major arc. Let $\overset{\frown}{AXB}$ be a semicircle. Then $m\overset{\frown}{AXB} = 180$.

Statement	*Reason*
(1) $m\overset{\frown}{AXB} = 180$	(1) Why ?
(2) $m\overset{\frown}{BYD} = m\angle BPD = 180 - m\angle APD$	(2) Why ?
(3) $m\overset{\frown}{AXB} + m\overset{\frown}{BYD} = 180 + 180 - m\angle APD = 360 - m\angle APD$	(3) Statements 1 and 2
(4) $m\overset{\frown}{ABD} = 360 - m\angle APD$	(4) $\overset{\frown}{ABD}$ is a major arc.
(5) $m\overset{\frown}{AXB} + m\overset{\frown}{BYD} = m\overset{\frown}{ABD}.$	(5) Statements 3 and 4.

Fifth Case. When either $\overset{\frown}{AXB}$ or $\overset{\frown}{BYD}$ is a major arc. Let $\overset{\frown}{AXB}$ be a major arc then $\overset{\frown}{BYD}$ will be a minor arc.

Statement	*Reason*
(1) D lies on B-side of $\overleftrightarrow{AP}$	(1) $\overset{\frown}{AXB}$ is a major arc

(2) D lies on A-side of $\overleftrightarrow{BP}$	(2) $\overset{\frown}{BXA}$ is a major arc

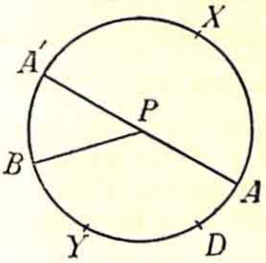


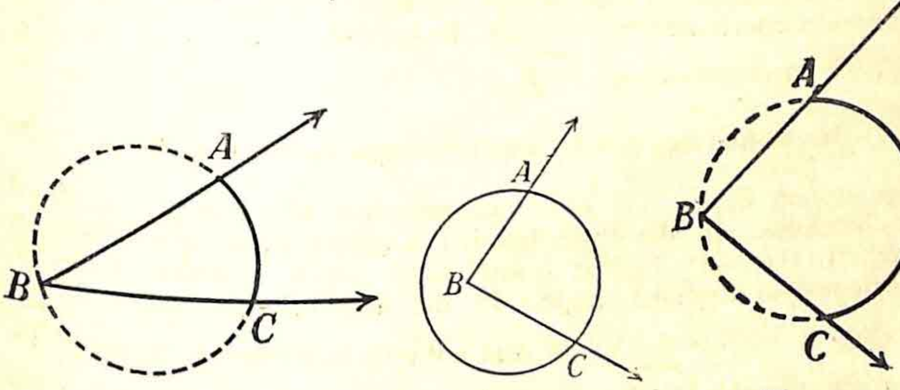
Fig 279.

(3) D lies in the interior of $\angle APB$	(3) Statements 1 and 2
(4) $m\angle APB = m\angle APD + m\angle BPD$	(4) Statement 3
(5) $m\angle APD = 360 - m\,\overset{\frown}{ABD}$	(5) $\overset{\frown}{ABD}$ is a major arc
(6) $m\angle BPD = m\overset{\frown}{BYD}$	(6) $\overset{\frown}{BYD}$ is a minor arc
(7) $m\angle APB = 360 - m\overset{\frown}{AXB}$	(7) $\overset{\frown}{AXB}$ is a major arc
(8) $360 - m\overset{\frown}{AXB} = 360 - m\overset{\frown}{ABD} + m\overset{\frown}{BYD}$	(8) Statements 5, 6, 7
(9) $-m\overset{\frown}{AXB} = -m\overset{\frown}{ABD} + m\overset{\frown}{BYD}$	(9) Statement 8
(10) $m\overset{\frown}{ABD} = m\overset{\frown}{AXB} + m\overset{\frown}{BYD}$	(10) Statement 9.

The proof of this theorem is rather lengthy and complicated and we could have taken it as a Postulate or accepted the result of theorem without any proof.

Angles Inscribed in Arcs

Besides the study of central angles that help us to define degree measure of arcs, it is important to know about inscribed arcs as many

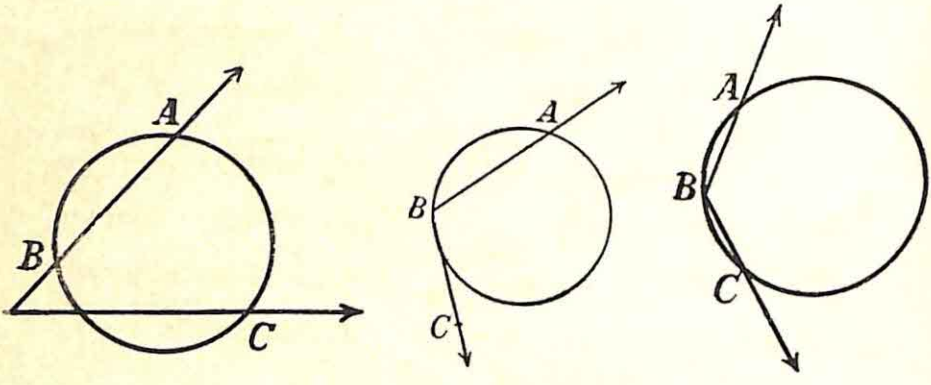


I II III

Fig 280.

important properties of circles involve the notion of inscribed angles.

In figure I of Fig. 280, $\angle ABC$ has been inscribed in a major arc $\overset{\frown}{ABC}$. In figure III, $\angle ABC$ has been inscribed in a semicircle $\overset{\frown}{ABC}$. In fiigure VI, $\angle ABC$ has been inscribed in a minor arc $\overset{\frown}{ABC}$. In these cases we find that the two ends of the arc lie on the two sides of the angle. The vertex of the angle is a point of the arc but it is not one of the end points of the arc. The figures IV, V and II donot show inscribed angles of the arcs.



IV V VI

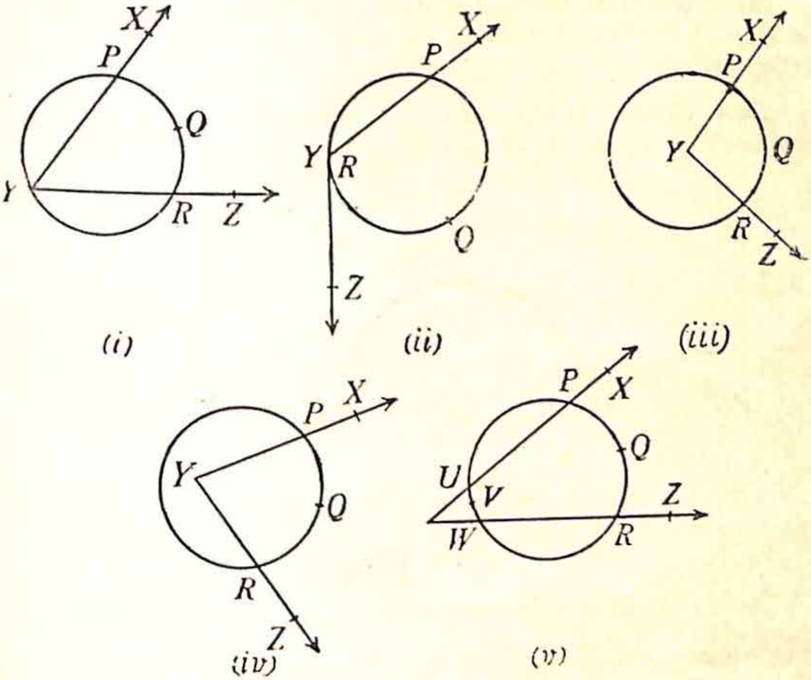
Fig. 281.

We define an inscribed angle of an arc as follows :—

Definition. XV. (15) An angle is said to be an inscribed angle of the arc iff (*i*) each side of the angle contains an end point of the arc. (*ii*) The vertex of the angle is an interior point (not an end point) of the arc.

Another notion with which we should be familiar with is that of an arc intercepted by an angle. In each of the figures of Fig. 282. $\angle XYZ$ intercepts an arc $\overset{\frown}{PQR}$.

In the first fig., it is an inscribed angle intercepting the arc. In the second figure $\overrightarrow{YZ}$ is a tangent while $\overrightarrow{YX}$ contains a chord of the circle. In the third fig., it is a central angle intercepting an arc. In the fourth fig., it is any angle which is neither a central angle nor an inscribed angle. In the fifth fig., the vertex of the angle is outside the circle and not only it intercepts the arc $\overset{\frown}{PQR}$ but also the arc $\overset{\frown}{UVW}$. From these figures we observe that each side of the angle contains one end point of the arc and also excepting its end points, the arc lies in the interior of the angle. We define an arc intercepted by an angle as follows.



(i) (ii) (iii)

(iv) (v)

Fig. 282.

Definition XV. (16) An angle is said to intercept an arc iff (*i*) Each end point of the arc is a point of the angle (*ii*) Each side of the angle contains atleast one end point of the arc (*iii*) Except for its end points, each point of the arc lies in the interior of the angle.

It is clear that each inscribed angle intercepts a unique arc on any given circle. The reason why we have introduced these concepts will be clear from the observation that we can describe the measure of an inscribed angle in terms of the degree measure of its inscribed arc (but how ?) and also there can be a lot of inscribed angles which may intercept a given arc on a given circle (and we are interested in knowing how these angles are related to each other). For this purpose, we prove the following theorems.

Theorem XV (14). In a circle, the measure of an inscribed angle is half the angle measure (degree measure) of its intercepted arc.

In order to prove the theorem, we have to prove the following three cases :

(1) When one side of an angle contains a diameter of the circle.

(2) When the end points of intercepted arc are on the opposite

side of the ray containing the diameter through the vertex of inscribed circle.

(3) When both the end points of intercepted arc, lie on the same side of the ray containing the diameter through the vertex of intercepted arc.

First Case. When one side of the angle contains a diameter of the circle.

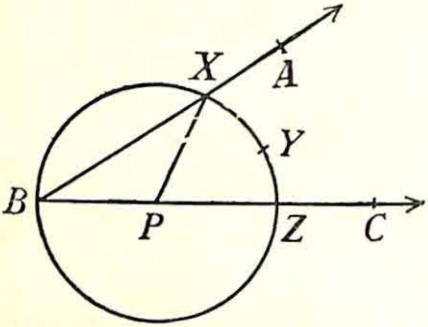


Fig. 283.

Given : A circle with centre P. An inscribed angle $\angle ABC$ intercepting an arc $\overset{\frown}{XYZ}$. The side $\overline{BC}$ of the angle contains P.

To prove that : Measure of $\angle ABC$ is half the angle measure of $\overset{\frown}{XYZ}$ or $m\angle ABC = \frac{1}{2}\, m\,\overset{\frown}{XYZ}$.

Proof.

Statement	*Reason*
(1) $m\angle XPZ = m\angle ABC + m\angle BXP$	(1) Strong form of exterior angle theorem.
(2) $\overline{PX} \cong \overline{PB}$	(2) Radii of the same circle
(3) $\triangle PBX$ is isosceles	(3) Statement 2
(4) $\angle ABC \cong \angle BXP$	(4) Statement 3
(5) $m\angle XPZ = 2m\angle ABC$	(5) Statements 1 and 4
(6) $m\angle XPC = m\,\overset{\frown}{XYZ}$	(6) Definition of degree measure of an arc
(7) $m\,\overset{\frown}{XYZ} = 2m\angle ABC$	(7) Statements 5 and 6
(8) $m\angle ABC = \frac{1}{2} m\overset{\frown}{XYZ}$	(8) Statement 7.

Second Case. When the end points of intercepted arc are on the opposite side of the ray containing the diameter through the vertex of inscribed angle.

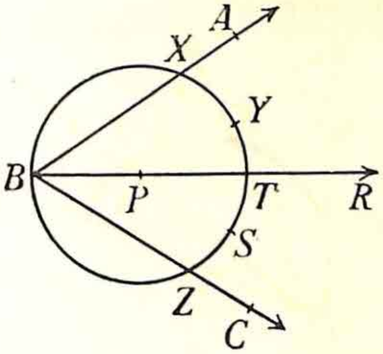


Fig. 284.

Given : A circle with centre P. An inscribed angle $\angle ABC$ intercepting arc $\overset{\frown}{XYZ}$. The end points X, Z of $\overset{\frown}{XYZ}$ lie on the opposite side of the ray $\overrightarrow{BR}$ which contains P and a point T of the circle.

To prove that : $m\angle ABC = \frac{1}{2} m\overset{\frown}{XYZ}$.

Proof.

Statement	*Reason*
(1) $m\angle ABR = \frac{1}{2} m\overset{\frown}{XYT}$, if Y is an interior part of $\overset{\frown}{XT}$	(1) First part of the theorem
(2) $m\angle RBC = \frac{1}{2} m\overset{\frown}{TSZ}$ if S is an interior part of $\overset{\frown}{TZ}$	(2) First part of the theorem
(3) $\overrightarrow{BR}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$	(3) Hypothesis. Plane Separation Postulate
(4) $m\angle ABC = m\angle ABR + m\angle RBC$	(4) Statement 3
(5) $m\overset{\frown}{XYT} + m\overset{\frown}{TSZ} = m\overset{\frown}{XYZ}$	(5) As $\overset{\frown}{XYZ}$ and $\overset{\frown}{TSZ}$ have only T in common. Theorem XV (13)
(6) $m\angle ABC = m\angle ABR + m\angle RBC$ $= \frac{1}{2} m\overset{\frown}{XYT} + \frac{1}{2} m\overset{\frown}{TSZ}$ $= \frac{1}{2}(m\overset{\frown}{XYT} + m\overset{\frown}{TSZ})$ $= \frac{1}{2} m\overset{\frown}{XYZ}$	(6) Statements 4, 1, 2, 5

Third Case. When both the end points of the intercepted arc lie on the same side of the ray containing the diameter through the vertex of the intercepted arc.

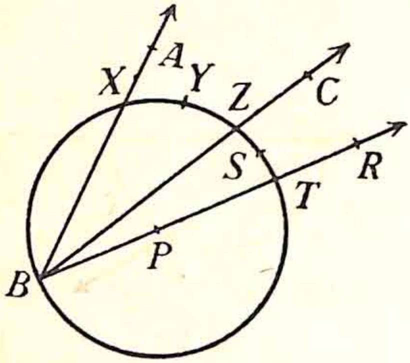


Fig. 285.

Given : A circle with centre P. An inscribed angle $\angle ABC$ intercepting the arc $\overparen{XYZ}$ whose end points X, Z lie on the same side of the ray $\overrightarrow{XR}$ which contains P and a point T of the circle.

To prove that : $m\angle ABC=\frac{1}{2}m\overparen{XYZ}$.

Proof.

Statement	*Reason*
(1) $\overrightarrow{BC}$ is between $\overrightarrow{BR}$ and $\overrightarrow{BA}$	(1) Hypothesis. For definiteness we assume that $\overrightarrow{BC}$ is between $\overrightarrow{BR}$ and $\overrightarrow{BA}$
(2) $m\angle ABR=m\angle ABC+m\angle CBR$	(2) Statement 1
(3) $m\angle ABR=\frac{1}{2}m\overparen{XZT}$	(3) First part of the theorem
(4) $m\angle CBR=\frac{1}{2}m\overparen{ZST}$ if S is interior point of $\overparen{ZT}$	(4) First part of the theorem
(5) $m\overparen{XZT}=m\overparen{XYZ}+m\overparen{ZST}$	(5) Theorem XV (13)
(6) $\frac{1}{2}m\overparen{XZT}=m\angle ABC+\frac{1}{2}m\overparen{ZST}$	(6) Statements 2, 3, 4 on substitution

(7) $\frac{1}{2}(m\overset{\frown}{XYZ}+m\overset{\frown}{ZST})$ $=m\angle ABC+\frac{1}{2}m\overset{\frown}{ZST}$	(7) Statements 5 and 6 on substitution
(8) $\frac{1}{2}m\overset{\frown}{XYZ}=m\angle ABC$	(8) Statement 7
(9) $m\angle ABC=\frac{1}{2}m\overset{\frown}{XYZ}$	(9) Statement 8.

The immediate corollaries to this theorem are :

(1) Angles inscribed in the same arc are congruent.

(2) An angle inscribed in a semicircle is a right angle.

We can prove these with the help of the theorem XV (14) as we know that in any arc we can inscribe as many angles as we want to and measure of each of these angles would be equal to half the degree measure of corresponding arc. Hence all these angles will be congruent. Also the measure of angle inscribed in a semicircle will be half the degree measure of corresponding semicircle and as the degree measure of a semicircle is 180, the measure of the angle inscribed in a semicircle is 90. However we prove these as theorems :

Theorem XV (15). Angles inscribed in the same arc of a circle are congruent and conversely :

Given : An arc $\overset{\frown}{XYZ}$ of a circle C with centre P

A set S of angles $\{\angle XLZ, \angle XMZ, \angle XNZ...\}$ inscribed in the arc $\overset{\frown}{XYZ}$

To prove that : The measure of each angle of the set S is same.

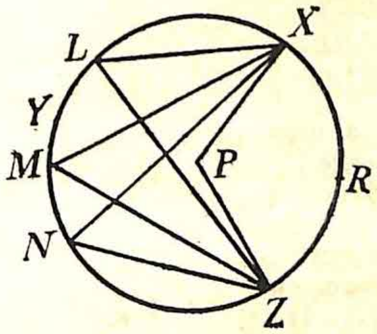


Fig. 286.

Proof. Let R be a point on the arc corresponding to $\overset{\frown}{XYZ}$ $(R \notin \overset{\frown}{XYZ})$. Now measure of each of angles of set S is half the degree measure of $\overset{\frown}{XRZ}$ which corresponds to $\overset{\frown}{XYZ}$ (Theorem XV (14). Hence the measure of each of angles of set S is the same.

Theorem XV (16). (*Converse to theorem XV (15)*. If a line segment joining two points subtends congruent angles on two other points on the same side of it, then the four points lie on a circle.

Given : A line l containing a line segment $\overline{AB}$ dividing a plane π into two half planes H_1 and H_2. Also two points, $C \in H_1$ and $D \in H_1$ such that $\angle ACB \cong \angle ADB$,

To prove that : A, B, C, D lie on the same circle.

Proof.

Statement	*Reason*
(1) There is one and only one circle through A, B and C	(1) Theorem XV (1)

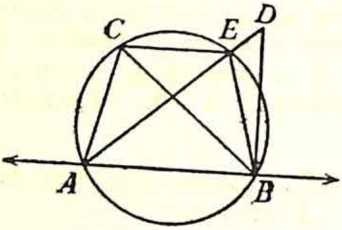


Fig. 287.

If the circle through A, B, C also contains D, then there is nothing further to prove. Assume that the circle through A, B, C does not contain D. Let the circle contain a point E of $\overline{AD}$ such that A—E—D (or A—D—E) (why should it contain E ?). Let us first take up the case A—E—D. Then :

(2) $\angle ACB \cong \angle AEB$	(2) Theorem XV (15)
(3) $\angle ACB \cong \angle ADB$	(3) Given
(4) $\angle AEB \cong \angle ADB$	(4) Statements 2 and 3.

Now statement 4 leads us to a contradiction as in $\triangle BED$, Exterior $\angle AEB >$ opposite interior $\angle ADB$ (Weak form of exterior angle theorem). Hence the assumption that the circle through A, B, C does not pass through D has to be abandoned. If we take up the case A—D—E then we arrive at the result that $\angle ADB \cong \angle AEB$ which also leads to a contradiction as the measure of an exterior angle of a triangle cannot be equal to the measure of opposite interior angle (why ?)

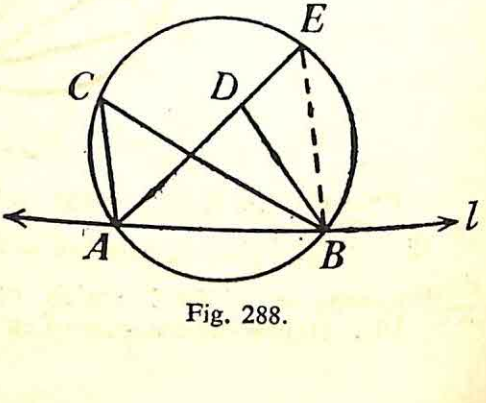


Fig. 288.

Theorem XV (17). The angle in a semicircle is a right angle.

Given : A circle C with centre P. Also its semicircle $\widehat{XYZ}$. An angle $\angle XLZ$ is inscribed in $\widehat{XYZ}$.

To prove that : $m\angle XLZ = 90$.

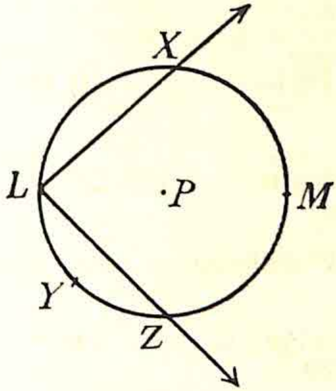


Fig. 289.

Proof. Let M be a point of the circle on the arc corresponding to $\widehat{XYZ}$. $(M \notin \widehat{XYZ})$ Then $m\angle XLZ = \frac{1}{2}\, m\widehat{XMZ} = \frac{1}{2} \times 180 = 90$ ($\widehat{XMZ}$ is also a semicircle. Why ?) Therefore $m\,\widehat{XMZ} = 180$.

Definition XV (17). (*Quadrilateral inscribed in a circle*). A quadrilateral is said to be inscribed in a circle if all its vertices lie on a circle.

Theorem XV (18). The opposite angles of any quadrilateral inscribed in a circle are supplementary and conversely.

(*A quadrilateral inscribed in a circle is called a cyclic quadrilateral*).

Given : A quadrilateral ABCD inscribed in a circle whose centre is P.

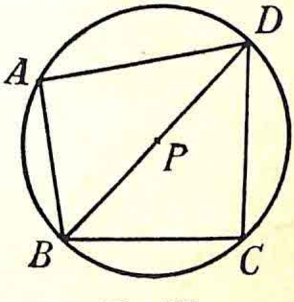


Fig. 290.

To prove that : (*i*) $m\angle A+m\angle C=180$.

(*ii*) $m\angle B+m\angle D=180$.

Proof: Let us discuss its proof in two cases :

Case First. When either $\overline{AC}$ or $\overline{BD}$ is a diameter of a circle. For definiteness let $\overline{BD}$ pass through P (a diameter).

Statement	*Reason*
(1) $m\angle A=\frac{1}{2}\ m\ \overset{\frown}{DCB}=\frac{1}{2}\times 180=90$	(1) $\overset{\frown}{DCB}$ is a semicircle
(2) $m\angle C=\frac{1}{2}\ m\ \overset{\frown}{DAB}=\frac{1}{2}\times 180=90$	(2) $\overset{\frown}{DAB}$ is a semicircle
(3) $m\angle A+m\angle C=90+90=\angle 180$	(3) Statements 1 and 2
(4) $m\angle A+m\angle B+m\angle C+m\angle D=360$	(4) Why ?
(5) $m\angle B+m\angle D=180$	(5) Statements 3 and 4 on substitution.

Case Second. When either of $\overset{\frown}{DAB}$, $\overset{\frown}{DCB}$ is a major arc. For definiteness let $\overset{\frown}{DAB}$ be a major arc so that $\overset{\frown}{DCB}$ is the corresponding minor arc.

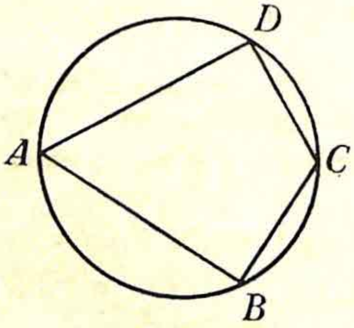


Fig. 291.

Statement	*Reason*
(1) $m\angle A=\frac{1}{2}\ m\ \overset{\frown}{DCB}$	(1) Why ?
(2) $m\angle C=\frac{1}{2}\ m\ \overset{\frown}{DAB}=\frac{1}{2}(360-m\ \overset{\frown}{DCB})$	(2) Why ?

(3) $m\angle A + m\angle C = \frac{1}{2}\, m\, \widehat{DCB}$ (3) Why ?

$+\frac{1}{2}\,(360 - m\,\widehat{DCB})$

$= \frac{1}{2}\, m\, \widehat{DCB} + 180 - \frac{1}{2} m\widehat{DCB}$

$= 180$

Converse to the theorem. If opposite angles of a quadrilateral are supplementary then the quadrilateral is inscribed in a circle.

Given : A quadrilateral ABCD such that : (1) $m\angle A + m\angle C = 180$ and (2) $m\angle B + m\angle D = 180$.

To prove that : Quadrilateral ABCD is inscribed in a circle.

Proof. There is one and only one circle through A, B, C (why ?) If this circle also contains D, then there is nothing further to prove.

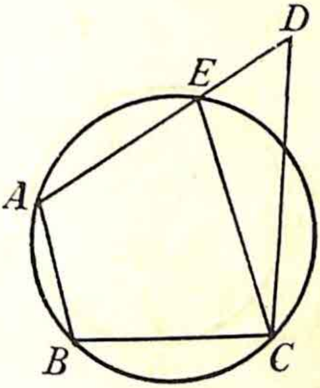


Fig. 292.

Let us assume that it does not contain D. Let it contain a point E of $\overline{AD}$ such that A$-$E$-$D (or A$-$D$-$E) (why ?) Let us first take up the case A$-$E$-$D.

Statement	*Reason*
(1) $m\angle B + m\angle D = 180$	(1) Why ? Given
(2) $m\angle B + m\angle CEA = 180$	(2) Why ?
(3) $m\angle CEA = m\angle D$	(3) Statements 1 and 2
(4) $m\angle CEA > m\angle D$	(4) Weak form of exterior angle theorem
(5) Statements 3 and 4 are contradictory to each other	(5) (Why ?)

Hence the assumption that the circle through A, B, C does not contain D has to be abandoned. If we take A—D—E, we will once again arrive at contradictory results.

Theorem XV (19). Given an angle with its vertex on a circle and formed by a secant ray and a tangent ray. The measure of the angle is half the measure of intercepted arc and conversely.

Given : A circle with centre P. Also an angle $\angle ABD$ whose vertex B is at the circle and which is formed by a secant ray $\overrightarrow{BA}$ and a tangent ray $\overrightarrow{BD}$.

To prove that : $m\angle ABD = \frac{1}{2}\, m\, \overset{\frown}{AEB}$ where E is an interior point on $\overset{\frown}{AB}$.

Proof. (*We assume that $\angle ABD$ is acute*)

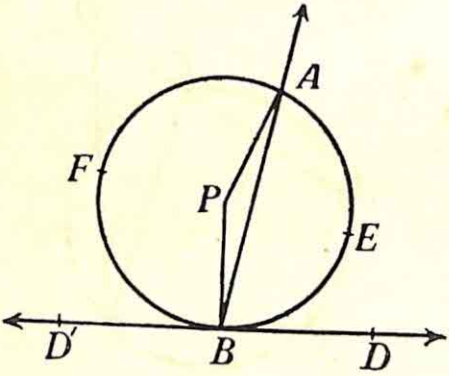


Fig. 293.

Statement	*Reason*
(1) $m\angle PBD = 90$	(1) Theorem XV (8)
(2) $\angle PAB \cong \angle PBA$	(2) APB is isosceles triangle
(3) $m\angle PBD = m\angle PBA + m\angle ABD$	(3) $\overrightarrow{BA}$ is between $\overrightarrow{BP}$ and $\overrightarrow{BD}$ as we have assumed that $\angle ABD$ is acute
(4) $90 = m\angle PBA + m\angle ABD$	(4) Statements 1 and 3
(5) $m\angle APB + m\angle PAB + m\angle PBA = 180$	(5) Angle-sum property of a triangle
(6) $m\angle APB + 2m\angle PBA = 180$	(6) Statement 2.
(7) $\frac{1}{2}\, m\angle APB + m\angle PBA = 90$	(7) Statement 6

(8) $\frac{1}{2} m\angle APB + m\angle PBA = m\angle PBA + m\angle ABD$	(8) Statements 4 and 7. If $a=b$ and $c=b$, then $a=c$
(9) $\frac{1}{2} m\angle APB = m\angle ABD$	(9) Statement 8. On applying concellation law.
(10) $m\angle ABD = \frac{1}{2} m\angle APB$	(10) Statement 9
(11) $m\angle ABD = \frac{1}{2} m\,\widehat{AEB}$	(11) Why ?

We can prove the case when the angle between the secant ray and tangent ray is obtuse by observing that in the figure of this theorem $\overline{DBD'}$ is a line and $\overrightarrow{BD'}$ is a tangent ray and $\angle ABD'$ is an obtuse angle. As $\angle ABD$ and $\angle ABD'$ form a linear pair, therefore $m\angle ABD' = 180 - m\angle ABD = 180 - \frac{1}{2} m\,\widehat{AEB} = \frac{1}{2}\,(360 - m\,\widehat{AEB})$ $= \frac{1}{2}\, m\,\widehat{AFB}$ where $\widehat{AFB}$ is the major arc corresponding to minor arc $\widehat{AEB}$. (According to our definition, the degree measure of a major arc equals 360 minus the degree measure of corresponding minor arc).

Converse to the theorem. Given an angle with vertex on the circle and formed by a secant ray and another ray which does not have any other point common with the circle excepting the vertex and such that the measure of the angle is half the measure of intercepted arc. Prove that the ray forming the angle, other than the secant ray is tangent to the circle.

Given . A circle with centre P. Also given a point B on the circle, a secant ray $\overrightarrow{BA}$ and other ray $\overrightarrow{BT}$. $\overrightarrow{BT}$ does not have any other point common with the circle excepting B. Also $m\angle ABT = \frac{1}{2} m\widehat{AEB}$ where E is an interior point of minor arc $\widehat{AB}$.

To prove that : $\overrightarrow{BT}$ is a tangent ray.

Proof. Assume that $\angle ABT$ is acute.

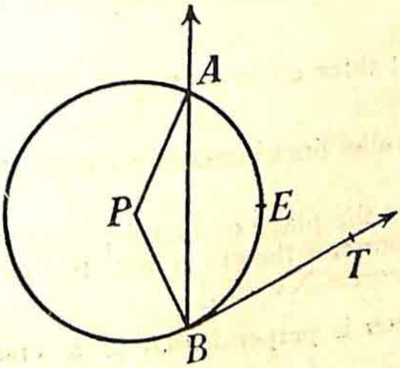


Fig. 294.

Statement	*Reason*
(1) $m\angle ABT=\frac{1}{2}\, m\, \overset{\frown}{AEB}$	(1) Given
(2) $m\angle ABT=\frac{1}{2}\, m\angle APB$	(2) $m\overset{\frown}{AEB}=m\angle APB$ (why ?)
(3) $\angle PAB\cong\angle PBA$	(3) Why ?
(4) $\frac{1}{2}\, m\angle APB+\frac{1}{2}\, m\angle PAB$ $+\frac{1}{2}\, m\angle PBA=90$	(4) Why ?
(5) $\frac{1}{2}\, m\angle APB+m\angle PBA=90$	(5) Statements 3 and 4
(6) $m\angle ABT+m\angle PBA=90$	(6) Statements 2 and 5
(7) $m\angle PBT=90$	(7) Statement 6 since $\overrightarrow{BA}$ is between $\overrightarrow{BP}$ and $\overrightarrow{BT}$ (why ?)
(8) $\overrightarrow{BT}$ is a tangent ray	(8) Since $\overline{BP}$ is radius. Statement 7.

Problem Set

A

1. If a line l does not intersect a circle C, then prove that the circle C lies in one of the half planes determined by l and does not have any points in common with the other half plane.

2. If S is a secant of the circle then prove that there are points of C in each of the half planes determined by S.

3. If t is a tangent to circle C at a point P, then prove that excepting P, all points of C lie on the same side of it.

4. Prove that every three non-collinear coplanar points lie on a circle.

5. Prove that three collinear points never lie on the same circle.

6. If two parallel lines intersect a circle, then they intercept congruent arcs.

7. If a line in the plane of a circle, intersects the interior of the circle, then it intersects the circle into two points. [Its proof is similar to that of theorem XV (5).]

.8 If a diameter is perpendicular to a chord, it bisects the chord and its arc.

[**Hint.** Prove that $\triangle APD \cong \triangle BPD$ by applying hyp. leg theorem. Hence $\overline{AD} \cong \overline{BD}$ and $\angle APD \cong \angle BPD \leftrightarrow m\ \widehat{AXE} = m\ \widehat{EYB}$]

9. Prove that if a diameter bisects a chord which is not a diameter, it is perpendicular to the chord and bisects the arc of the chord.

(Converse to the theorem of problem 8).

10. Prove that if a diameter bisects one of the parallel chords, it bisects the other chord.

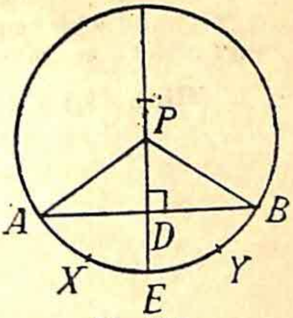


Fig. 295.

11. Given a circle with centre P. AB is a diameter and AC is any other chord from A. $\overleftrightarrow{CD}$ is tangent at C and $\overleftrightarrow{PD} \parallel \overleftrightarrow{AC}$. Prove that $\overleftrightarrow{DB}$ is tangent at B.

[**Hint.** (1) $x = y$ (why ?) (2) $x = l$ (why ?)
(3) $u = v$ (why ?) (4) $v + l = 90$ (why ?)
(5) $u + w = 90$ (why ?) (6) $l = w$ (why ?)

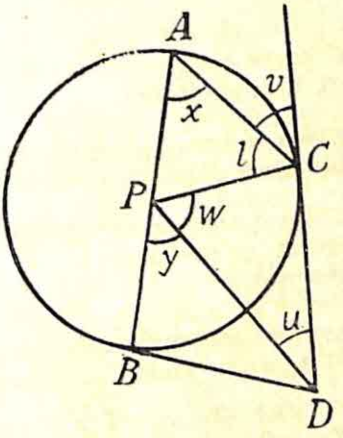


Fig. 296.

(7) $x = w$ (why ?) (8) $w = y$ (why ?)
(9) $\triangle PCD \cong \triangle PBD$ (why ?)
(10) $\angle PCD \cong \angle PBD$ (why ?)
(11) $m\ \angle PBD = 90$ (why ?)]

12. Prove that if a chord is perpendicular bisector of another chord in the same circle, it is a diameter of the circle.

13. Prove that if a chord of a circle is perpendicular to one of two parallel chords of the circle and bisects the other chord, it is a diameter of the circle.

14. Given : two concentric circles C_1, C_2 with radii r_1, r_2 ($r_1 > r_2$) and with centre P. Prove that all chords of C_1 which are tangent to C_2 are bisected at the point of contact.

[**Hint.** $\overline{PD}$ is $\perp$ $\overline{AB}$. Hence $\overline{AD} \cong \overline{BD}$ why ?]

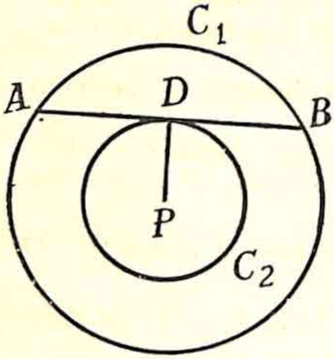


Fig. 297.

15. Prove that if a diameter of a circle bisects each of two chords which are not diameters, the chords are parallel to each other.

16. Prove that if two chords of a circle are drawn from a point of the circle and make congruent angles with the radius drawn to the point, the chords are congruent.

17. Prove that two points of a circle, each equidistant from two other points of the circle determine a diameter of the circle.

18. Prove that if two congruent chords which are not diameters intersect a diameter at the same point, they make congruent angles with it and conversely.

19. Prove that the perpendicular bisectors of sides of a polygon inscribed in a circle, meet in a point.

20. If two congruent chords of a circle intersect, the segments of one are respectively congruent to segments of the other.

21. Prove that two distinct circles cannot have more than two points in common.

22. Prove that a circle and a line cannot have more than two points in common.

23. Given a pentagon ABCDE. If the quadrilaterals ABCD and BCDE are cyclic, prove that the pentagon is also cyclic.

[**Hint.** Circles through A, B, C. D and B, C, D, E have three points B, C, D in common. Therefore the circle through BCD, passes through A & E].

B

1. Show that if two circles are tangent to the same line at the same point, the line of centres passes through the point of contact.

2. If two chords are unequal in measure, their distances from the centre of the circle are unequal in measure in opposite order.

[**Hint.** Given $m\ \overline{AB} > m\ \overline{CD}$, P is the centre. $\overline{PE} \perp \overline{DC}$ and $\overline{PF} \perp \overline{AB}$.

$m\ BF = \frac{1}{2}\ m\ \overline{AB}$ and $m\ \overline{EC} = \frac{1}{2}\ m\ \overline{CD}$ (why ?)

$m\ \overline{BF} > m\ \overline{EC}$ (why ?) $\overline{PC} \cong PB$.

$\triangle$ PEC and $\triangle$ PFB are right triangles $\therefore$ $PB^2 = PF^2 + FB^2$

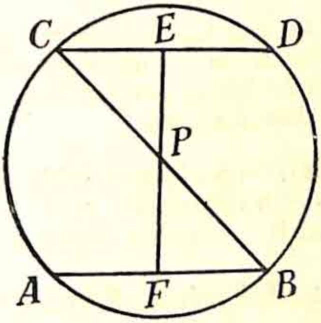


Fig. 298.

and $PC^2 = PE^2 + EC^2$ Also $PB^2 = PC^2$ (why?)

Hence $PF^2 + FB^2 = PE^2 + EC^2$. Now $FB^2 > EC^2$.

Hence $PF^2 < PE^2$

Hence $PF < PE$.]

3. Prove the converse to the theorem given in problem 2. Its statement is—If two chords are unequally distant from the centre of the circle, the measure of the chords are unequal in opposite order.

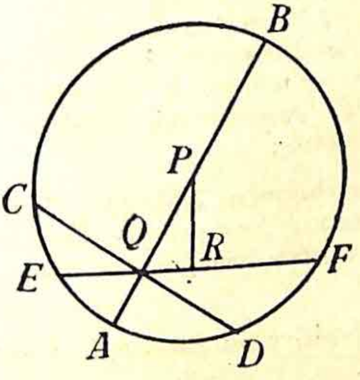


Fig. 299.

4. Prove that the shortest chord which can be drawn through a point inside a circle is a chord which is perpendicular to the diameter through the point.

[**Hint.** In Fig. 299. Let $\overline{PR}\perp\overline{EF}$, $\overline{CD}\perp\overline{AB}$. In the right triangle PQR, $PR<PQ$ (why ?) $\therefore EF>CD$.]

5. Given a circle with centre P. $\overline{APB}$ is a diameter of the circle. $\overline{AC}$ and $\overline{BD}$ are any chords such that $m\,\overline{AC}>\overline{BD}$, $\overline{BC}\perp\overline{AC}$ and $\overline{AD}\perp\overline{BD}$ (A, B, C, D are points on the circle). Prove that $m\,\overline{BC}<m\,\overline{AD}$.

6. Given a circle with centre P. $\overline{AB}$ is any diameter and C, D are any other points on the circle but on the same side of AB. Also $AC=BD=AP$. Prove that $CD=BP$.

7. Two circles C_1 and C_2 are externally tangent to same line l at T. There is a point R on l. $\overline{RA}$ is tangent to C_1 where A is the point of tangency and RB is tangent to C_2 where B is also point of tangency. Prove that $\overline{RA}\cong\overline{RB}$.

8. A, B are points on a circle C with centre P such that A, B are not the end points of a diameter. t_1 is tangent to C at A and t_2 is tangent to C at B. t_1 and t_2 have a point D in common. $\overleftrightarrow{DP}$ contains two points of the circle, E and F such that D—E—P—F. Prove that $AF=BT$.

Euclid again.

Yet another instance where Euclid overlooked the necessity of stating a postulate or proving a theorem is provided by Euclid's first proposition. This proposition is about the construction of an equilateral triangle on a given line segment as base. He carried out this construction in the same way as a school child does it—by striking arcs of the appropriate radius from the ends of line segments as centres and then joining the points of intersection of the arcs to the ends of line segment. But Euclid never stated any axiom or proved a theorem regarding his assumption that the two arcs intersect. None of Euclid's axioms deals with the conditions under which two circles intersect. He postulates conditions when two lines will intersect or will not intersect but he failed to do so in case of two circles. To remove this weakness we prove what is known as triangle theorem :

The Triangular theorem XV (20). Given three positive real numbers a, b, c. If each of these numbers is less than the sum of the other two, then there exists a triangle, the measure of whose sides is a, b, c.

Given : Three positive numbers a, b, c such that

(*i*) $a<b+c$ (*ii*) $b<a+c$ (*iii*) $c<a+b$

To prove that : There exists a △, the measure of which sides is a, b, c.

Proof. Without any loss of generality, we can assume that

$$a \geqslant b \geqslant c.$$

Statement	*Reason*
(1) There is a segment $\overline{BC}$ whose measure is a	(1) Segment construction theorem
(2) On $\overline{BC}$, there is an arbitrary point D such that B—D—C. Measure of $\overline{BD}$ is a real number x such that $x<a$	(2) Segment construction theorem

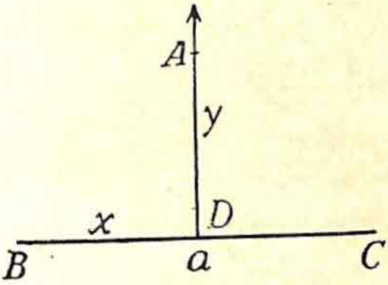


Fig. 300.

(3) At D, there is a unique line l, $\perp \overline{BC}$ (and coplanar with it)	(3) Why ?
(4) On l, there is an arbitrary point A such that $\overline{DA}=y$.	(4) Why ?

Now can A be one of the vertices of a △ whose other two vertices could be B and C and the measures of whose sides are a, b, c ? Of course while thinking like this we are assuming the existence of such a triangle, but once we have fixed up three points which are non-collinear, we may be able to check whether these can serve as the vertices of a △ that we are looking for. So we assume that △ ABC exists formed by the vertices that we have located and $\overline{AD}$ is perpendicular segment from A to $\overline{BC}$ and B—D—C as we have already assumed that $\overline{BD}$ is the longest side of △ ABC. Also measure of $\overline{AB}=c$ and measure of $\overline{AC}=b$ Since $\overline{BD}=x$ and B—D—C, $\overline{DC}=a-x$.

(5) $AB^2=BD^2+AD^2$	(5) Pythagoras theorem. Since ABD is a right triangle.

(6) $c^2=x^2+y^2$	(6) Statement 5, on substituting the values of $\overline{AB}$, BD and $\overline{AD}$
(7) $AC^2=DC^2+AD^2$	(7) Pythagoras theorem
(8) $b^2=(a-x)^2+y^2$	(8) Reasoning same as for Statement 6.
(9) $c^2=x^2+y^2,\ b^2=(a-x)^2+y^2$ $\leftrightarrow c^2-b^2=x^2+y^2-(a-x)^2-y^2$ $\leftrightarrow c^2-b^2=x^2-(a-x)^2$ $\leftrightarrow c^2-b^2=(x+a-x)\times(x-a+x)$ $\leftrightarrow c^2-b^2=a(2x-a)$ $\leftrightarrow c^2-b^2=2ax-a^2$ $\leftrightarrow 2ax=c^2-b^2+a^2$ $\leftrightarrow x=\dfrac{a^2+c^2-b^2}{2a}$ Also $y^2=c^2-x^2$ $\leftrightarrow y=\sqrt{c^2-x^2}$	(9) Solving simultaneously
(10) $c^2=x^2+y^2,\ b^2=(a-x)^2+y^2$ $\leftrightarrow x=\dfrac{a^2+c^2-b^2}{2a}$ and $y=\sqrt{c^2-x^2}$	(10) Statement 9

Now will x and y exist subject to what is given to us and the assumptions that we have made ? We have assumed $a \geqslant b \geqslant c$. So $a^2 \geqslant b^2$ and $c^2 \geqslant 0$. We want to dismiss the case when $a=b=c$ and $c=0$.

So we take $a^2 \geqslant b^2$ and $c^2>0$ So $x=\dfrac{a^2+c^2-b^2}{2a}$ $\leftrightarrow x=\dfrac{(a^2-b^2)+c^2}{2a} \leftrightarrow x>0$. So there exists a positive number x.

Now $y=\sqrt{c^2-x^2}$. Now for the existence of y as a positive number we must show that

$$c^2-x^2>0 \quad \leftrightarrow \quad (c-x)>0$$

$$c^2>x^2 \quad \leftrightarrow \quad c>x \quad \leftrightarrow \quad (c-x)>0$$

Now $c-x=c-\dfrac{a^2+c^2-b^2}{2a}$

$$\leftrightarrow \quad c-x=\frac{2ac-a^2-c^2+b^2}{2a}$$

$$\leftrightarrow \quad c-x=\frac{b^2-(a-c)^2}{2a}$$

$$\leftrightarrow \quad c-x=\frac{(b+a-c)(b-a+c)}{2a}$$

$$\leftrightarrow \quad c-x=\frac{(b+a-c)(b+c-a)}{2a}$$

Now we are given that $a<b+c$ and $c<b+a$, therefore $(b+a-c)>0$ and $(b+c-a)>0$. Hence $\frac{(b+a-c)(b+c-a)}{2a}>0$

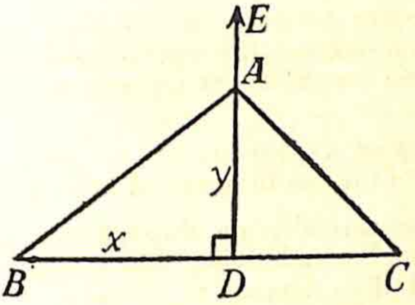


Fig. 301.

$\leftrightarrow c-x>0 \leftrightarrow c>x$ Hence $y>0$.

We can construct our triangle by proceeding in the following way :

Let $\overline{BC}$ be line segment of length a. Let D be a point in $\overline{BC}$ such that B $-$ D $-$ C and $m\overline{BD}=x=\frac{a^2+c^2-b^2}{2a}$.

Let $\overrightarrow{DE}$ be the ray beginning at D and perpendicular to $\overline{BC}$ and let A be the point of $\overrightarrow{DE}$ such that $\overline{DA}=y=\sqrt{c^2-x^2}$. Then $\triangle$ABC is the required $\triangle$. We can verify that $m\overline{AB}=c$ and $m\overline{AC}=b$.

Since $AB^2=AD^2+BD^2$

$$\leftrightarrow \quad AB^2=y^2+x^2 \quad \leftrightarrow \quad AB^2=c^2-x^2+x^2$$

$$\leftrightarrow \quad AB^2=c^2 \quad \rightarrow \quad AB=c$$

and $AC^2=DC^2+AD^2 \rightarrow AC^2=(a-x)^2+y^2$.

$$\rightarrow \quad AC^2=a^2+x^2-2ax+c^2-x^2$$

$$\rightarrow \quad AC^2=a^2-2ax+c^2$$

$$\rightarrow \quad AC^2=a^2+c^2-2ax$$

$$\leftrightarrow \quad AC^2=a^2+c^2-2a\times\frac{a^2+c^2-b^2}{2a}$$

$$\leftrightarrow \quad AC^2=a^2+c^2-a^2-c^2+b^2$$

$$\sim \quad AC^2=b^2$$

$$\leftrightarrow \quad AC=b.$$

The Two Circle Theorem XV (21). If c_1 and c_2 are any two coplanar circles with centres P_1 and P_2 and radii r_1 and r_2 respectively. If P_1P_2, the line segment determined by the centres of c_1 and c_2 equals p, then :—

(1) If $r+r_2>p$, $r_1+p>r_2$ and $r_2+p>r_1$ the circles have two and only two points in common (or the circles intersect at two and only two points).

(2) If $r_1+r_2>p$ or $r_1+p=r_2$ or $r_2+p=r_1$ (only one of these conditions need to be satisfied) the circles have one and only one point in common and the circles are tangents to each other at that point.

(3) If $r_1+r_2<p$ or $<r_1+p<r_2$ or $r_2+p<r_1$ the circles do not intersect (only one of the conditions need to be satisfied).

Proof. If we carefully go through the statement of this theorem, we find that the underlying idea is when a $\triangle ABC$ would exist if the measures of its sides are r_1, r_2 and p. If the triangle can be constructed then according to the statement of the theorem, the circles will intersect at two points. If the triangles cannot be constructed, then under certain conditions, the circles will be tangents to each other and under certain other conditions, the circles will not intersect at all.

To prove this theorem, we will have to consider the following cases :

(*a*) As we are given two distinct circles, hence $P_1=P_2$ and $r_1=r_2$ simultaneously cannot hold. If $P_1=P_2$ and $r_1\neq r_2$ then $p=0$ and then the set of inequalities given in number (1) become.

$$r_1+r_2>0,\ r_1>r_2 \text{ and } r_2>r_1.$$

It is not possible to have both $r_1>r_2$ and $r_2>r_1$. Hence the circle will not intersect. We know that if $P_1=P_2$ and $r_1\neq r_2$ then c_1 and c_2 will be concentric circles and they do not intersect.

In this case, equalities in number (2) become $r_1+r_2=0$, $r_1=r_2$ and $r_2=r_1$.

We have already stated that the conditions $r_1=r_2$ or $r_2=r_1$ and $r_1+r_2=0$ are not possible. Hence the circles will not be tangent to each other.

Inequalities in (3) become

$$r_1+r_2<0 \text{ or } r_1<r_2 \text{ or } r_2<1$$

It is not possible to have $r_1+r_2<0$ (why ?) We can have only either $r_1<r_2$ or $r_2<r_1$. In either case the circles will not intersect. Hence the assertion of the theorem is verified if $P_1=P_2$ and $r_1\neq r_2$.

Now let us take up the case when $P_1\neq P_2$ and $r_1\neq r_2$. First let us take up the case when circles intersect at A. Let $\overline{AD}$ be the perpendicular segment from A to $\overline{P_1P_2}$ where D can have either of the following positions :

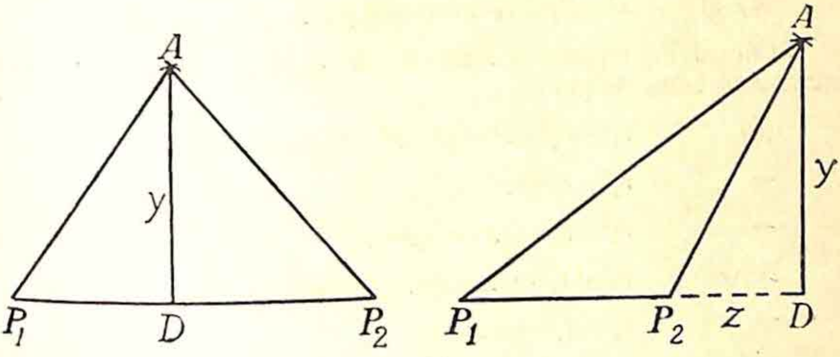


Fig. 302.

(1) P_1-D-P_2 (2) P_1-P_2-D (3) $D-P_1-P_2$

(4) D coincides with P_1 (5) D coincides with P_2.

If $AD=y$, then $(P_1D)^2+(AD)^2=(AP_1)^2$ ...(1)

and $(P_1D)^2+(AD)^2=(AP_2)^2$ ...(2)

Let $m(\overline{P_1D})=x$ and $m(\overline{P_2D})=z$

and $AP_1=r_1$ and $AP_2=r_2$

Equation (1) becomes

$$x^2+y^2=r_1^2 \quad ...(3)$$

and equation (2) becomes.

$$z^2+y^2=r_2^2 \quad ...(4)$$

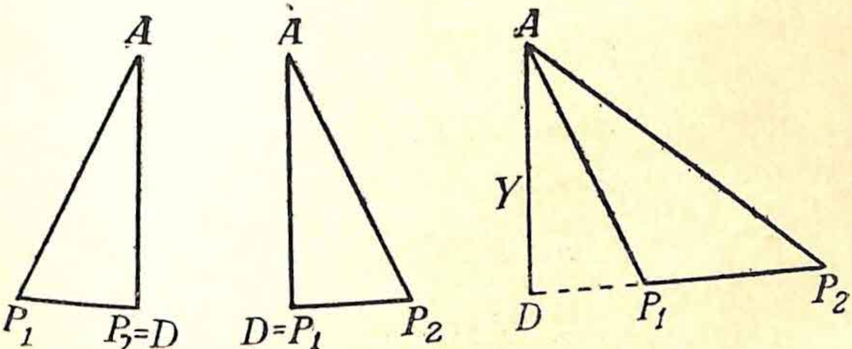


Fig. 303.

Now let us discuss all the five cases.

(*a*) If P_1-D-P_2 then $x+z=p$

(*b*) If P_1-P_2-D then $p+z=x$

(*c*) If $D-P_1-P_2$ then $x+p=z$

(*d*) If $D=P_1$, then $x=0$ and $p=z$

(*e*) If $D=P_2$ then $z=0$ and $p=x$

On solving equations 3 and 4 for x, z in each of the above mentioned cases we get :

(*a*) $r_1^2-r_2^2=x^2+y^2-z^2-y^2$

→ $r_1^2-r_2^2=x^2-z^2$

→ $r_1^2-r_2^2=(x+z)(x-z)$

Now $r_1^2-r_2^2=(x+z)(x-z)$ and $x+z=p$

→ $r_1^2-r_2^2=p(x-z)$

→ $$x-z=\frac{r_1^2-r_2^2}{p}.$$

So $$x-z=\frac{r_1^2-r_2^2}{p} \text{ and } x+z=p$$

→ $$x=\frac{r_1^2-r_2^2+p^2}{2p}$$

and $$z=\frac{p^2-r_1^2+r_2^2}{2p}.$$

(*b*) $r_1^2-r_2^2=(x+z)(x-z)$

and $p+z=x$

→ $r_1^2-r_2^2=(x+z)(x-z)$

and $x-z=p$

→ $r_1^2-r_2^2=(x+z)p$

→ $$x+z=\frac{r_1^2-r_2^2}{p}$$

Now $$x+z=\frac{r_1^2-r_2^2}{p}$$

and $x-z=p$

→ $$x=\frac{r_1^2-r_2^2+p^2}{2p}$$

or $$z=\frac{r_1^2-r_2^2-p^2}{2p}$$

(c) $r_1^2-{r^2}_2=(x+z)(x-z)$

and $x+p=z$

$\rightarrow$ $r_1^2-r_2^2=(x+z)(x-z)$

and $x-z=-p$

$\rightarrow$ $r_1^2-r_2^2=(x+z)(-p)$

$\leftrightarrow$ $x+z=\dfrac{r_1^2-r_2^2}{-p}.$

Now $x+z=\dfrac{r_1^2-r_2^2}{-p}$

and $x-z=-p$

$\rightarrow$ $x=\dfrac{r_1^2-{r^2}_2+p^2}{-2p}$

and $z=\dfrac{r_1^2-r_2^2-p^2}{-2p}.$

(d) $r_1^2-r_2^2=(x+z)(x-z)$

and $x=0$

and $p=z$

$\rightarrow$ $r_1^2-r_2^2=-p^2\ ;\ x=0,\ z=p.$

(e) $r_1^2-{r^2}_2=(x+z)(x-z)$

and $z=0,\ p=x$

$\rightarrow$ $x=p,\ z=0$

and $r_1^2-r_2^2=p^2$

Now if we solve $x^2+y^2=r_1^2$

and $z^2+y^2=r_2^2$

for y^2 with the help of the values of x and z given in each of (*a*), (*b*), (*c*), (*d*), (*e*) we get that

$$y^2=\frac{(r_1+r_2+p)(r_1+r_2-p)(r_1+p-r_2)(r_2+p-r_1)}{4p^2} \qquad \ldots(5)$$

In every case we will find this very value for y^2. Let us verify in one case.

$x^2+y^2=r_1^2$

and $x=\dfrac{r_1^2-r_2^2+p^2}{2p}$ in first case

$\rightarrow$ $\left(\dfrac{r_1^2-r_2^2+p^2}{2p}\right)^2+y^2=r_1^2$

$\rightarrow$ $y^2=r_1^2-\left(\dfrac{r_1^2-r_2^2+p^2}{2p}\right)^2$

$$\rightarrow \qquad y^2=\left(r_1+\frac{r_1^2-r_2^2+p^2}{2p}\right)\left(r_1-\frac{r_1^2-r_2^2+p^2}{2p}\right)$$

$$\rightarrow \qquad y^2=\frac{(r_1^2-r_2^2+p^2+2pr_1)(r_2^2-r_1^2-p^2+2pr_1)}{4p^2}$$

$$\rightarrow \qquad y^2=\frac{[(p^2+r_1^2+2pr_1)-r_2^2][r_2^2-(r_1^2+p^2-2pr_1)]}{4p^2}$$

$$\rightarrow \qquad y^2=\frac{[(p+r_1)^2-(r_2)^2][(r_2^2)-(r_1-p)^2]}{4p^2}$$

$$\rightarrow \qquad y^2=\frac{(p+r_1+r_2)(p+r_1-r_2)(r_2+r_1-p)(r_2-r_1+p)}{4p^2}\,.$$

Which is same as we have earlier stated.

Similarly if we take $z^2+y^2=r_2^2$

and $z=\dfrac{p^2-r_1^2+r_2^2}{p^2}$ we will get the same value of y^2. We can also verify in other cases that the distance y from A to $\overline{P_1P_2}$ is necessarily given by (5).

Now since x and z are non-negative, then it follows from the values of x and z in (a), (b), (c), (d) (e) that we must have respectively,

(a) $r_1^2+p^2>r_2^2$ and $p^2+r_2^2>r_1^2$

(b) $r_1^2+p^2>r_2^2$ and $r_1^2>p^2+r_2^2$

(c) $r_1^2+p^2<r_2^2$ and $r_1^2<r_2^2+p^2$.

(Since denominator is negative in each case)

(d) $r_1^2+p^2=r_2^2$ and since $r_2^2>r_1^2$ in this case $r_2^2+p^2>r_1^2$

(e) $r_2^2+p^2=r_1^2$ and since $r_1^2>r_2^2$ in this case $r_1^2+p^2>r_2^2$.

Now three numbers r_1, r_2 and p must satisfy one and only one of the following relations.

(6) (a) $r_1^2+p^2>r_2^2$ $\quad(b)$ $r_1^2+p^2=r_2^2$ $\quad(c)$ $r_1^2+p^2<r_2^2$

(7) (a) $r_2^2+p^2>r_1^2$ $\quad(b)$ $r_2^2+p^2=r_1^2$ $\quad(c)$ $r_2^2+p^2<r_1^2$

Now $6(c)$ and $6(b)$ imply $r_1^2>r_2^2$

and $7(c)$ and $7(b)$ imply $r_2^2>r_1^2$

So either 6 (c) or 6 (b) holds and then neither 7 (a) nor 7 (b) will hold and *vice versa*.

Now 6 (a) (b) (c) and 7 (a) (b) (c) give us nine possible pairs of conditions out of which only following can hold :

(6*a*, 7*a*), (6*a*, 7*c*), (6*c*, 7*a*), (6*b*, 7*a*) and (7*b*, 6*a*).

Moreover, one and only one pair of these conditions must hold. Of course these are respectively the conditions that are needed in

a, b, c, d and e. Thus either one of these holds at a time and x, z and the point D are uniquely determined implying thereby that if two coplanar circles intersect then the common points lie on the unique line lying in the plane of the circles which is perpendicular to P_1P_2. Since we have already proved a theorem stating that a line cannot have more than two points in common with a circle (which ?), so two circles can intersect in at the most two points. Moreover the conditions given in the first part of the theorem are sufficient as we know that

$$y^2 = \frac{(r_1+r_2+p)(r_1+r_2-p)(r_1+p-r_2)(r_2+p-r_1)}{4p^2}$$

and if $r_1+r_2>p$, $r_1+p>r_2$ and $r_2+p>r_1$ then y^2 is positive and $y=AD$ exists. Also there is a unique pair of values x, z satisfying equations 3 and 4 and hence a unique point D is derermined. Moreover there will be two points A_1 and A_2 such that $A_1D=A_2D=y$. Hence the assertion made in the first part of the theorem.

If the conditions in part 2 of the theorem are satisfied, that is, if $r_1+r_2=p$ or $r_1+p=r_2$ or $r_2+p=r_1$ then $y^2=0$ or $y=0$ and hence $y=AD=0$. $y=0$ satisfies the equation 3 and 4 as then we get $x^2=r_1^2$ and $z^2=r_2^2$ or $x=r_1$ and $z=r_2$. In this case the point D is common to both the circles and as then we get $x^2=r_1^2$ and $z^2=r_2^2$ which implies that circles are tangents to each other (why ? which Theorem ?)

Thus the assertion of part 2 of the theorem is satisfied. Moreover if we consider the conditions given in part 3 of the theorem, *viz.*,

$$r_1+r_2<p \quad \text{or} \quad r_1+p<r_2 \quad \text{or} \quad r_2+p<r_1$$

then we find that if either one of these is satisfied then

$$y^2 = \frac{(r_1+r_2+p)\ (r_1+r_2-p)\ (r_1+p-r_2)\ (r_2+p-r_1)}{4p^2}$$

or y^2 is negative as one of the four factors in the numerator is negative and hence y is neither positive nor zero. Therefore two circles cannot have any portion common. Thus part 3 of the theorem is verified.

This is how we establish this theorem.

A problem that we want to consider now is : How many tangent lines we can draw to a circle from a point exterior to it ? To get an answer to this, let us prove the following theorem :

Theorem. Given a circle C and a point Q lying in its plane but in its exterior. There are exactly two lines which are tangent to C and contain Q.

Given. A circle with centre P and a coplanar point Q lying in the exterior of the circle.

To prove that. There are exactly two lines tangents to C and containing the point Q.

Proof. Let r be the radius of the circle. Let $m\overline{PQ}=a$. Let M be the mid-point of the line segment $\overline{PQ}$, so $m\overline{PM}=\frac{a}{2}=r'$ (say). Now will a circle with M as centre and radius r' intersect the circle C ? If so in how many points ? To get answers to these questions, let us apply the two circle theorem. Here (*i*) $r=r$ (*ii*) $r'=\frac{a}{2}$ (*iii*) $p=\frac{1}{2}\,m(\overline{PM})=\frac{a}{2}$

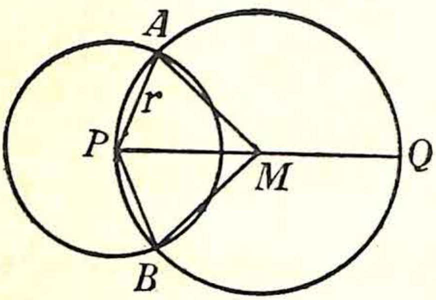


Fig. 304.

Therefore

(*a*) $r+r'=r+\frac{a}{2}>p$

(*b*) $r+p=r+\frac{a}{2}>r'$

(*c*) $r'+p=\frac{a}{2}+\frac{a}{2}=a>r$ (why ? Is it because Q is in the exterior of the circle ?)

Hence a circle with centre M and radius r' will intersect the circle C in two points A and B (say) and now these points will be on the opposite sides of the line of centres. Now $m\angle PAQ=90$ (Angle in a semi-circle as $\overline{PB}$ is the diameter). Similarly $m\angle PBQ=90$ (why ?) Therefore $\overleftrightarrow{AQ}$ and $\overleftrightarrow{BQ}$ are tangent lines to C at A and B (why ?) So there exist at least two tangent lines to C and containing Q. Can we have more than two ? No (why not ?)

Hence there exist two tangent lines to C and containing Q.

We have talked about the measure of a line or 'distance between the points.' When we talked about the distance between two points A and B, intitutively did, we think of the distance transversed on an arc containing these points ? Certainly no. Had we imagined it like this then there would have not been one answer to this but many more—as many as we want to.

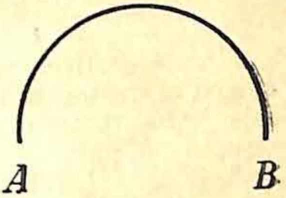


Fig. 305.

Certainly we have talked about the measure of an arc but that was more precisely about the degree measure of an arc ? With its help we can compare the measure of two arcs on the same circle or on congruent circles ? What about two arcs lying on two different circles which are not congruent to each other ? Can we develop some formulae to find the length measure of an arc ?

Intitutively we can think of an arc $\widehat{AYB}$ as union of $n+1$ chords $\overline{AY_1}$, $\overline{Y_1Y_2}$, $\overline{Y_2Y_3}$ ......$\overline{Y_{n-1}Y_n}+Y_nB$

Where Y_1, Y_2...are interior points of the arc.

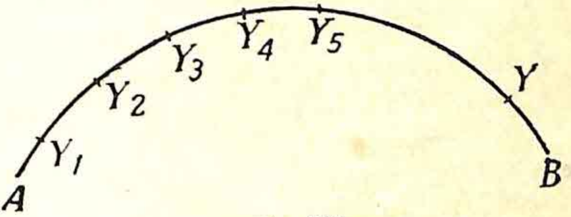


Fig. 306.

We can say that this approximation is 'very close' to $\widehat{AYB}$ if our n is large enough. As n becomes larger and larger, the length of each of chords $\overline{AY_1}$, $\overline{Y_1Y_2}$......becomes smaller and smaller and then $S_{n+1}=AY_1+Y_1Y_2......$ will approach a limit L (say). So we can conveniently define the length of $\widehat{AYB}$ in terms of L. We can similarly define the

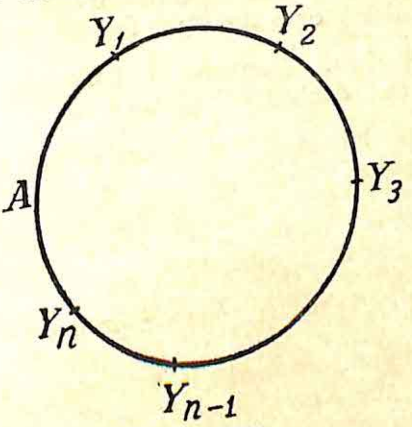


Fig. 307.

circumference of a circle in terms of the limit of sum S_{n+1} of the lengths of chords

$$\overline{AY_1}, \overline{Y_1Y_2}, \overline{Y_2Y_3}, \ldots\ldots\ldots\ldots, \overline{Y_{n-1}Y_n}, \overline{Y_nA}$$

So intitutively we think of circumference of a circle in terms of sum of the measures of sides of an inscribed polygon of $n+1$ sides. Just to simplify the matters we decide to think the circumference of a circle in terms of a regular polygon of 2^n-sides; 4, 8, 16, 32, 64, 128...2^n-sides. It is appropriate to take 2^n as then we can easily make n as large as we please. Ofcourse as n becomes larger and larger, the length of each of sides of the inscribed polygon gets smaller and smaller. So accordingly we make the following definition.

Circumference of a circle. The circumference of a circle is the limit of perimeter of a regular polygon of 2^n sides inscribed in a circle as n gets larger and larger.

Length of an arc. If A is a vertex of a regular polygon of 2^n-sides inscribed in a circle containing an arc $\widehat{AYB}$ and if $Y_1, Y_2, Y_3 \ldots\ldots Y_r$ are the other vertices of the inscribed polygon which are also the points of $\widehat{AYB}$ then length of $\widehat{AYB}$ is the limit L of the sum $S_{n+1}=AY_1+Y_1Y_2\ldots\ldots Y_nB$ as n gets larger and larger.

Theorem XV (22). The circumference C, of any circle of radius arc equals $2\pi r$. In other words $C=2\pi r$.

To establish this formula, we have to prove the following two theorems:

Theorem XV (23). The perimeter of a regular polygon of 2^n-sides inscribed in a given circle approaches a limit as n becomes infinite (larger and larger).

Theorem XV (24). The ratio of the circumference of a circle to the length of its diameter is the same for all circles.

Theorem XV (23). To prove that the perimeter of a regular polygon of 2^n sides approaches a limit then we have to prove that:

(1) Perimeter increases monotonically *i.e.* as n increases perimeter also increases.

(2) Perimeter is bounded from above or perimeter has an upper bound which has a definite limit.

Proof. To prove the first part we proceed as follows:

Let C be the given circle with centre P. Let $Y_1, Y_2, Y_3\ldots\ldots$ $\ldots\ldots Y_r, Y_{r+1}\ldots\ldots$be the vertices of inscribed polygon of 2^n sides. The perimeter of this polygon will be $S_n=2^n(Y_rY_{r+1})$.

Now we have to show that the perimeter of an inscribed polygon of 2^{n+1} sides is greater than the perimeter of an inscribed polygon of 2^n sides.

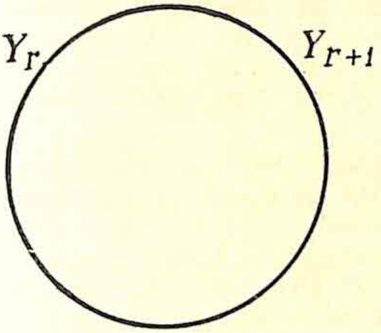


Fig. 308.

Now to get a polygon of 2^{n+1} sides we have to locate a point Y_r' between every two consecutive vertices Y_r, Y_{r+1} of the inscribed polygon of 2^n sides such that $\overrightarrow{PY_r'}$ is the angle bisector of $\angle Y_rPY_{r+1}$.

Now the perimeter of regular polygon of 2^{n+1} sides will be given by $S_{n+1}=2^n\,(Y_rY'_r+Y_r'Y_{r+1})$

Now $Y_rY_{r+1}\,Y'_r$ is a $\triangle$ hence $Y_rY'_r+Y_r'Y_{r+1}>Y_rY_{r+1}$

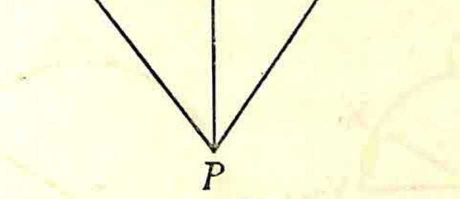


Fig. 309.

Hence clearly $S_{n+1}>S_n$.

as $\quad 2^n(Y_rY'_r+Y'_rY_{r+1})>2^n(Y_rY_{r+1})$

Hence the perimeter of the inscribed polygon increases monotonically as n gets larger and larger.

Now to show that the perimeter has an upper bound, we consider the circumscribed polygon of 2^n sides whose sides are tangents to the given circle at the vertices of inscribed polygon. By a similar argument we can show that (*i*) the perimeter of circumscribed polygon decreases monotonically as n gets larger and larger and

(*ii*) Perimeter of a circumscribed polygon of 2^n sides would always be greater than the perimeter of inscribed polygon of 2^n sides. Of course in both these we will make use of triangle inequality. (The sums of the measures of any two sides of a triangle is larger than the measure of the third side). Hence S_n approaches a limit.

Now we can say that the circumference C of a circle would be less than the perimeter of circumscribed polygon of 2^n sides as n gets larger and larger but more than the perimeter of inscribed polygon of 2^n sides as n gets larger and larger. Of course we are aided by the fact that while the perimeter of circumscribed polygon is monotonically decreasing that of inscribed polygon is monotonically increasing.

Theorem XV(24). The ratio of the circumference of a circle to the length of its diameter is the same for all circles.

Proof. Let us consider two circles C_1 and C_2 with centres P_1 and P_2 and with radii r_1 and r_2 respectively. Let Y_r and Y_{r+1} be any two consecutive vertices of a polygon of 2^n sides inscribed in circle C_1 and let Y'_r and Y'_{r+1} be any two consecutive vertices of a polygon of 2^n sides inscribed in C_2

Then $m\angle Y_rP_1Y_{r+1} = \frac{360}{2^n}$ (why ?)

Also $m\angle Y'_rP_2Y'_{r+1} = \frac{360}{2^n}$ (why ?)

Hence $\angle Y_rP_1Y_{r+1} \cong \angle Y'_rP_2Y'_{r+1}$

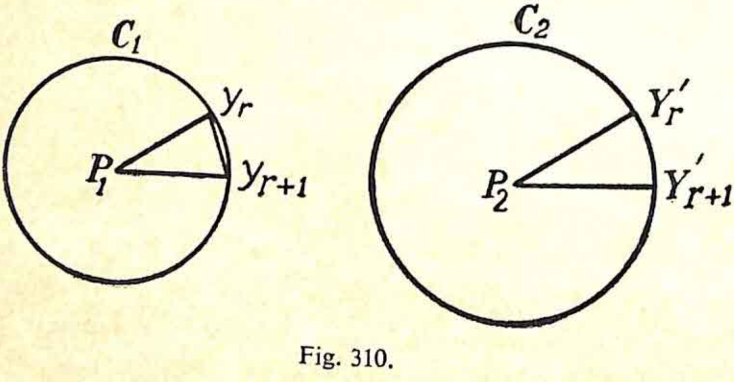


Fig. 310.

Hence 1—1 correspondence $P_1Y_rY_{r+1} \leftrightarrow P_2Y'_rY'_{r+1}$ is a similarity and $\triangle P_1Y_rY_{r+1} \sim \triangle P_2Y'_rY'_{r+1}$.

(S.A.S. similarity theorem. How ?)

Hence $$\frac{(Y_rY_{r+1})}{P_1Y_r} = \frac{Y'_rY'_{r+1}}{P_2Y'_r}$$

$$\leftrightarrow \qquad \frac{Y_rY_{r+1}}{r} = \frac{Y'_rY'_{r+1}}{r'}$$

$$\leftrightarrow \qquad \frac{2^n}{2} \times \frac{(Y_rY_{r+1})}{r} = \frac{2^n}{2} \times \frac{(Y'_rY'_{r+1})}{r'}$$

$$\leftrightarrow \qquad \frac{2^n(Y_rY_{r+1})}{2r} = \frac{2^n(Y'_rY'_{r+1})}{2r'}$$

$$\leftrightarrow \qquad \frac{S_n}{d} = \frac{S'_n}{d'} \qquad \ldots(i)$$

Where S_n is the perimeter of polygon of 2^n sides inscribed in C_1 and d is a diameter of C_1 and S'_n is the perimeter of polygon of 2^n sides inscribed in C_2 and d' is the diameter of C_2.

Now on taking limits of each of the sides of (i) as n becomes infinite, we get :

$$\underset{n \to \infty}{\mathrm{Lt}} \left(\frac{S_n}{d}\right) = \underset{n \to \infty}{\mathrm{Lt}} \left(\frac{S'_n}{d'}\right)$$

$$\leftrightarrow \qquad \frac{1}{d} \underset{n \to \infty}{\mathrm{Lt}} S_n = \frac{1}{d'} \underset{n \to \infty}{\mathrm{Lt}} (S'_n)$$

$$\leftrightarrow \qquad \frac{C}{d} = \frac{C'}{d'}$$

Where C is the circumference of C_1 and C' is the circumference of C_2 as by our definition.

$\underset{n \to \infty}{\mathrm{Lt}} S_n$ equals the circumference of C_1 and

$\underset{n \to \infty}{\mathrm{Lt}} S'_n$ equals the circumference of C_2.

Hence we prove that the ratio of circumference of a circle to its diameter is the same for all circles.

Now $\frac{C}{d}$ is noted by familiar symbol π which approximately equals 3·1614. Its more accurate value is 3·14159265358979. However value of π has been calculated even up to a thousand places of a decimal with the help of electronic computer. Therefore

$$\frac{C}{d} = \pi \leftrightarrow C = \pi d$$

or $$C = \pi . 2r \leftrightarrow C = 2\pi r.$$

Now to develop the concept of area of a circular region we have to make the following definitions,

Definition. The union of a circle and its interior is called a circular region.

Definition. The area of a circular region is the limit of area of a polygonal region bounded by a regular polygon of 2^n sides inscribed in a circle as n becomes infinite.

To establish this, we have to prove that :

(1) The area of a polygonal region bounded by a polygon of 2^n sides inscribed in a circle increases monotonically.

(2) The area of a polygonal region bounded by the circumscribed polygon of 2^n sides whose sides are tangents to the given circle at the vertices of inscribed polygon is always greater than the area of polygonal region bounded by the inscribed polygon but it decreases monotonically as n gets larger and larger.

We can prove these statements by employing the same technique as we did while proving $C=2\pi r$.

Now area of polygonal region is equal to $2^n \times$ Area of

$$\triangle PY_rY_{r+1}=2^n\times\tfrac{1}{2}\times PN\times Y_rY_{r+1}$$

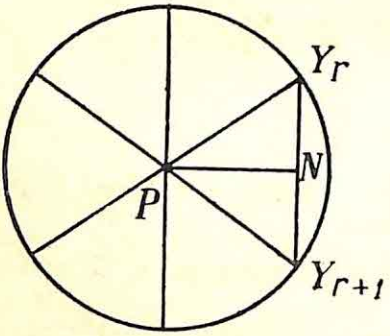


Fig. 311.

Where PN is perpendicular segment from P to Y_rY_{r+1}

$$\leftrightarrow \quad 2^n.\ \tfrac{1}{2}\ (PN)\left(\frac{2^n\times Y_rY_{r+1}}{2^n}\right)$$

$\leftrightarrow \quad \dfrac{2^n}{2^n}\ \tfrac{1}{2}\ (PN)\ S_n$ where S_n is the perimeter of inscribed polygon $(S_n=2^n\ .\ Y_rY_{r+1})$

$$\leftrightarrow \quad \tfrac{1}{2}\ (PN)\ S_n,$$

Now the area of a circle will be limit of $\tfrac{1}{2}\ (\overline{PN})\ S_n$ as n gets infinite or $$A = \lim_{n\to\infty} \tfrac{1}{2}(PN)\, S_n$$

$$\to \tfrac{1}{2} \lim_{n\to\infty} (PN) \times \lim_{n\to\infty} S_n.$$

$$\to \tfrac{1}{2}\, r\, C$$

Hence $$A = \tfrac{1}{2} r .\ 2\pi r = \pi r^2$$

Problem Set

1. Prove that the perimeter of a regular polygon of n sides circumscribed about a circle decreases monotonically as n gets larger and larger.

2. Prove that the perimeter of a regular polygon of n sides inscribed in a circle increases monotonically as n gets larger and larger.

3. Prove that area of polygonal region bounded by inscribed polygon increases monotonically.

4. Prove that the area of polygonal region bounded by circumscribed polygon decreases monotonically.

In this course our effort has been to integrate methods of Pure Geometry with that of Analytical Geometry: Accordingly we proved earlier some theorems by the method of Pure Geometry as well as with Analytical Geometry. Some of the theorems that we proved by both the methods are :

(1) Concurrency of medians of triangle.

(2) Concurrency of bisectors of angles of triangles.

(3) Measure of the line segment joining the middle points of two sides of a triangle is half the measure of the third side.

We can prove many more theorems of Pure Geometry with the help of the Analytical methods. We now prove one or two more.

Theorem. Prove that the line segment joining the middle points of two sides of a triangle is parallel to the third side.

Given **:** A $\triangle$ ABC, the co-ordinates of whose vertices are $A(x_1 y_1)$, $B(x_2 y_2)$ and $C(x_3\, y_3)$.

To prove that : $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$.

Proof. If D is the mid point of $\overline{AB}$ and E is the mid point of $\overline{AC}$, then the co-ordinates of D and E are

$$D\left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2}\right);\ E\left(\frac{x_1+x_3}{2},\ \frac{y_1+y_3}{2}\right)$$

Now the slope of $\overleftrightarrow{DE}$ will be

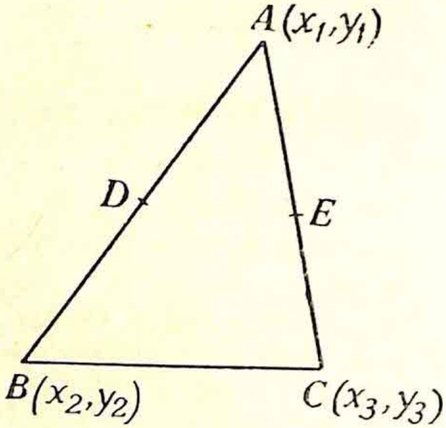


Fig. 312.

$$\frac{\frac{y_1+y_2}{2}-\frac{y_1+y_3}{2}}{\frac{x_1+x_2}{2}-\frac{x_1+x_3}{2}}=\frac{y_2-y_3}{x_2-x_3}$$

The slope of $\overleftrightarrow{BC}$ will be $\frac{y_2-y_3}{x_2-x_3}$

Hence $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$

Theorem. Prove that the diagonals of a square are perpendicular to each other.

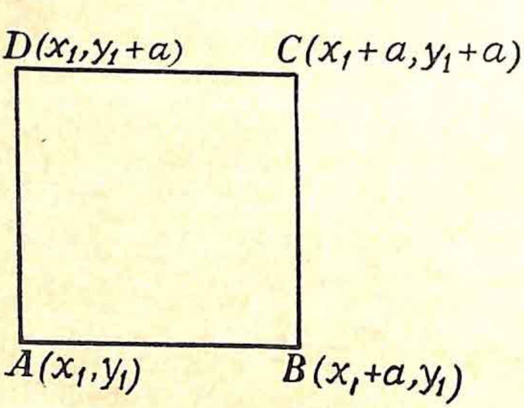


Fig. 313.

Given : A square ABCD, the measure of each of whose sides is a. If the co-ordinates of A are A(x_1, y_1), then those B, C, D will be :

$B(x_1+a, y_1)$; $C(x_1+a, y_1+a)$; $D(x_1, y_1+a)$ (why ?)

$$\text{Now slope of } \overleftrightarrow{AC} = \frac{y_1+a-y_1}{x_1+a-x_1} = 1 = m(\text{say})$$

$$\text{Slope of } \overleftrightarrow{BD} = \frac{y_1+a-y_1}{x_1-x_1-a} = \frac{a}{-a} = -1 = \frac{1}{m}$$

Hence $\overleftrightarrow{AC} \perp \overleftrightarrow{BD}$ as $m \times \frac{1}{m} = -1$

We can prove many more theorems involving figures consisting of lines, angles etc. etc. Let us now take up some about circles. To do this, we will first have to develop analytical theory about circles.

We have defined a circle as a set of all those points which lie in a given plane π and whose distance from a given point P in π is a given positive number r.

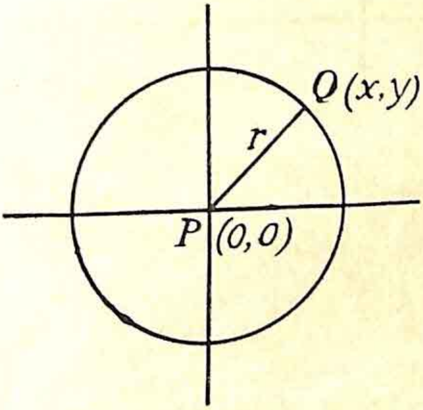


Fig. 314

Let us consider a circle whose radius is r and whose centre is at the origin. So we know that the centre of circle is P(0, 0). If $Q(x, y)$ is an arbitrary point on the circle then

$$PQ = \sqrt{(x-0)^2+(y-0)^2} \text{(why ?)}$$

$$= \sqrt{x^2+y^2}$$

Hence $\sqrt{x^2+y^2} = r$ or $x^2+y^2=r^2$

By reversing these steps we can show that if $x^2+y^2=r^2$, then $P(x, y)$ is a point on the circle whose distance from its centre is r. If we actually draw this circle and take an arbitrary point on it, we shall find that its co-ordinates will satisfy the equation $x^2+y^2=a^2$

Now let us consider the general case when the centre is at the point $P(h, k)$ and its radius is r.

Let $Q(x,y)$ be an arbitrary point on the circle. Now $PQ=r$. and $PQ=\sqrt{(x-h)^2+(y-k)^2}$ (why ?)

Hence $\sqrt{(x-h)^2+(y-k)^2}=r$

$\leftrightarrow \quad (x-h)^2+(y-k)^2=r^2$

$\leftrightarrow \quad x^2-2hx+h^2+y^2-2ky+k^2=r^2$

$\leftrightarrow \quad x^2+y^2-2hx-2ky+h^2+k^2-r^2=0$

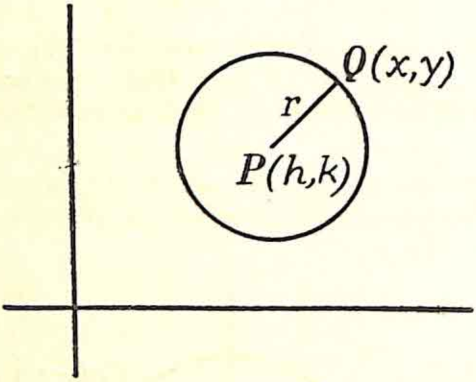


Fig. 315.

The standard equation of a circle is $(x-h)^2+(y-k)^2=r^2$ whose centre is $P(h, k)$ and whose radius is r. Of course we can write it in the form $x^2+y^2-2hx-2ky+h^2+k^2-r^2=0$ and then we can write it as

$x^2+y^2+2gx+2fy+c=0$ if we write

$-h=g$, $-k=f$ and $h^2+k^2-r^2=c$. This is called the general form of an equation of a circle.

Theorem. The graph of an equation of the form $x^2+y^2+2gx+2fy+c=0$ is (1) a circle (2) a point or (3) an empty set.

We have already established that every circle is graph of an equation of the form $x^2+y^2+2gx+2fy+c=0$. In other words we have established that the equation of an arbitrary circle will be of the form $x^2+y^2+2gx+2fy+c=0$

Is the converse to this also true ? In other words will the graph of an equation of the form $x^2+y^2+2gx+2fy+c=0$ always be a circle ?

Now the given equation is $x^2+y^2+2gx+2fy+c=0$ and we can write it as :

$x^2+y^2+2gx+2fy+c=0.$

$(x^2+2gx+g^2)+(y^2+2fy+f^2)=g^2+f^2-c$

$\leftrightarrow \quad (x+g)^2+(y+f)^2=(\sqrt{g^2+f^2-c})^2$

Now when we compare it with standard form of equation if a circle *viz.*, $(x-h)^2+(y-k)^2=r^2$

then we find that if $h=-g$, $k=-f$ and $r=\sqrt{g^2+f^2-c}$ then it will be a circle whose centre is $(-g,-f)$ and whose radius is $\sqrt{g^2+f^2-c}$. This implies that g^2+f^2-c should be a positive number or

$$g^2+f^2-c>0$$

Now what happens if $g^2+f^2-c=0$? In this case the given equation becomes $(x+g)^2+(y+f)^2=0$ and the graph of this equation will be a single point $(-g,-f)$. Now if $g^2+f^2-c<0$ which implies that the sum of two squares $(x+g)^2$, $(y+f)^2$ is a negative number g^2+f^2-c then

The given equation is not satisfied by the co-ordinates of any point. Then the graph of the given equation will be an empty set.

We, therefore, summarise this discussion as follows :

The graph of $x^2+y^2+2gx+2fy+c=0$ will be (1) a circle (2) a point or (3) an empty set, according as if g^2+f^2-c is positive, zero or negative.

Example 1. Find the equation of the circle whose centre is origin and whose radius is 4 units.

Solution. The required equation is

$$(x-0)^2+(y-0)^2=(4)^2$$

$$\leftrightarrow \qquad x^2+y^2=16,$$

Example 2. Find the equation of a circle whose centre is origin and which contains a point Q (6, 8).

Solution. Let the radius of the given circle be r. Then the equation of the circle is

$$(x-0)^2+(y-0)^2=r^2 \quad \leftrightarrow \quad x^2+y^2=r^2$$

Now as it contains the point (6, 8), hence

$$(6)^2+(8)^2=r^2 \qquad \text{or} \quad 100=r^2.$$

Hence the required equation is $x^2+y^2=100$

Can you do this problem through somewhat different approach ? Will not the radius be equal to $\sqrt{(6-0)^2+(8-0)^2}=\sqrt{100}$ $=10$ (why ?)

Example 3. Find the equation of a circle whose centre is P(1, 2) and whose radius is 5 units.

Solution. The required equation will be

$$(x-1)^2+(y-2)^2=(5)^2$$

$$\leftrightarrow \qquad (x-1)^2+(y-2)^2=25.$$

Example 4. Find the equation of a circle whose centre is P$(-3, -4)$ and which contains a point Q$(2, 3)$.

Solution. Now the radius of the circle will be

$$PQ = \sqrt{(-3-2)^2+(-4-3)^2}$$
$$= \sqrt{25+49} = \sqrt{74}$$

Hence the required equation will be $(x+3)^2+(y+4)^2=74$.

Alternately. Let the radius of the given circle be r. Hence the equation of the circle will be $(x+3)^2+(y+4)^2=r^2$. Now it contains the point Q$(2, 3)$. Hence $(2+3)^2+(3+4)^2=r^2$

$$25+49=r^2 \quad \leftrightarrow \quad 74=r^2$$

Example 5. Will the graph of equation $x^2+y^2-4x+6y+9=0$ be a circle ? If so, find its radius as well as the co-ordinates of its centre.

Solution. $$x^2+y^2-4x+6y+9=0$$
$$\leftrightarrow \quad (x^2-4x+4)+(y^2+6y+9)=4+9-9$$
$$\leftrightarrow \quad (x-2)^2+(y+3)^2=(2)^2.$$

Clearly the graph of the equation is a circle whose radius is 2 units and the co-ordinates of whose centre are $(2, -3)$,

We have already proved that the graph of the equation $x^2+y^2+2gx+2fy+c=0$ will be a circle if g^2+f^2-c is positive. This equation has three unknown quantities namely g, f and c and assuming that g^2+f^2-c is positive, we need three equations containing g, f, and c to find out the value of g, f and c to enable us to determine the equation of a circle. It implies that we should know the coordinates of any three points on the circle. We have already proved that three non-collinear points determine one and only one circle and the same is being indicated to us. Suppose we are given three points A$(3, 0)$; B$(-3, 0)$ and C$(0, 5)$. We can prove that these points are non-collinear. We will be led to the same conclusion if we come to graph these points. Will these determine a circle ? Let us assume that these points do determine a circle and its equation is

$$x^2+y^2+2gx+2fy+c=0$$

Then as the circle contains the points

A $(3, 0)$, B $(-3, 0)$ and C $(0, 5)$. Hence

$$(3)^2+(0)^2+2g(3)+2f(0)+c=0$$

and $$(-3)^2+(0)^2+2g(-3)+2f(0)+c=0$$

and $$(0)^2+(5)^2+2g(0)+2f(5)+c=0$$

On simplification we find that this set of equations is equivalent to the following set

$$6g+c+9=0 \qquad \ldots(1)$$

$$-6g+c+9=0 \qquad \ldots(2)$$

and

$$10f+c+25=0 \qquad \ldots(3)$$

On solving equations (1) and (2) simultaneously we get $2c+18=0$ or $c=-9$ and $g=0$. If $c=-9$ then $f=\frac{-16}{10}=\frac{-8}{5}$ on substituting the value of C in equation (3),

Hence the equation $x^2+y^2+2gx+2fy+c=0$

becomes

$$x^2+y^2+\frac{-16}{5}y-9=0$$

and we can write it as

$$(x-0)^2+\left(y-\frac{8}{5}\right)^2=9+\frac{64}{25}$$

$$\leftrightarrow \quad (x-0)^2+\left(y-\frac{8}{5}\right)^2=\frac{289}{25}$$

$$\leftrightarrow \quad (x-0)^2+\left(y-\frac{8}{5}\right)^2=\left(\frac{17}{5}\right)^2$$

Hence the centre of this circle is $\left(0, \frac{8}{5}\right)$

and its radius is $\frac{17}{5}$ units.

Example 6. Will A (1, 3) ; B (1, −2) and C (5, 3) determine a circle ? If so, find its radius and co-ordinates of its centre.

Solution. Let us assume that these points determine a circle and let its equation be

$$x^2+y^2+2gx+2fy+c=0,$$

As A (1, 3), B (1, −2) and C (5, 3) lie on this circle.

Hence $(1)^2+(3)^2+2g(1)+2f(3)+c=0$

and $(1)^2+(-2)^2+2g(1)+2f(-2)+c=0$

and $(5)^2+(3)^2+2g(5)+2f(3)+c=0$

Which on simplification become

$$2g+6f+c=-10 \qquad \ldots(1)$$

and

$$2g-4f+c=-5 \qquad \ldots(2)$$

and

$$10g+6f+c=-34 \qquad \ldots(3)$$

On eliminating 'c' from equations (1) and (2) and from (2) and (3) we get

$$10f=-5 \qquad \text{and} \qquad -8g-10f=29$$

Hence we get $f=-\frac{1}{2}$ and $\quad -8g=24$ *i.e.*, $g=-3$

Now as $\qquad 2g+6f+c=-10$

Hence $\qquad -6+(-3)+c=-10$

or $\qquad c=-1$

Hence $x^2+y^2+2gx+2fy+c=0$

becomes $\qquad x^2+y^2-6x-y-1=0$ which we can write as

$$(x^2-6x+9)+(y^2-y+\tfrac{1}{4})=9+\tfrac{1}{4}+1$$

or $\qquad (x-3)^2+(y-\frac{1}{2})^2=10\frac{1}{4}.$

Hence the coordinates of the centre of the circle are $(3, \frac{1}{2})$ and the radius is $\sqrt{10\frac{1}{4}}$.

Theorem. If the co-ordinates of the end points of a diameter of a circle $\overline{AB}$ are A (x_1, y_1) and B (x_2, y_2) then the equation of the

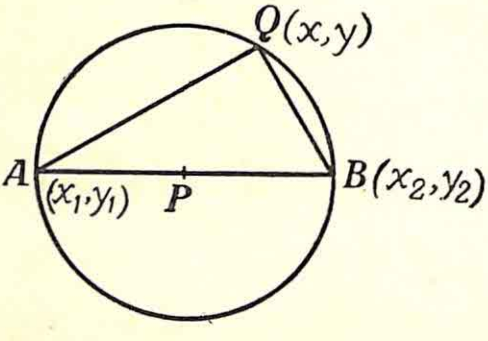


Fig. 316.

circle is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$

Given : The coordinates of the end points of diameter $\overline{AB}$ are A $(x_1\ y_1)$ and B (x_2, y_2)

To prove that : The equation of the circle will be

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

Proof. Let Q $(x_1\ y)$ be any arbitrary point on the circle. The centre of the circle will be mid point of $\overline{AB}$.

Hence the centre will be P $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$(How ?)

Also the radius r of the circle will be $\frac{1}{2}\overline{AB}$

Hence $\qquad r=\frac{1}{2}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

Therefore the equation of the circle is

$$\left(x-\frac{x_1+x_2}{2}\right)^2+\left(y-\frac{y_1+y_2}{2}\right)=\left(\tfrac{1}{2}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\right)^2$$

$$\leftrightarrow\quad x^2-x(x_1+x_2)+\frac{(x_1+x_2)^2}{4}+y^2-y(y_1+y_2)+\frac{(y_1+y_2)^2}{4}$$

$$=\tfrac{1}{4}[(x_1-x_2)^2+(y_1-y_2)^2]$$

$$\leftrightarrow\quad x^2-x(x_1+x_2)+x_1x_2+y^2-y(y_1+y_2)+y_1y_2=0$$

$$\leftrightarrow\quad (x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

Alternatively :—

$m\angle\text{AQB}=90$ (why ?)

Now the slope of $\overleftrightarrow{\text{QA}}$ is $\dfrac{y-y_1}{x-x_1}$

and the slope of $\overleftrightarrow{\text{QB}}$ is $\dfrac{y-y_2}{x-x_2}$

Hence $\dfrac{y-y_1}{x-x_1}\times\dfrac{y-y_2}{x-x_2}=-1$ $[m\angle\text{AQB}=90]$

or $(y-y_1)(y-y_2)=-1(x-x_1)(x-x_2)$

$\leftrightarrow\quad (y-y_1)(y-y_2)+(x-x_1)(x-x_2)=0$

Alternatively :—

$$(\text{QA})^2+(\text{QB})^2=(\text{AB})^2 \qquad \text{(why ?)}$$

$$[(x-x_1)^2+(y-y_2)^2]+[(x-x_2)^2+(y-y_2)^2]=(x_1-x_2)^2+(y_1-y_2)^2$$

$$\leftrightarrow\quad 2x^2+2y^2-2xx_1-2xx_2-2yy_1-2yy_2+x_1^2+x_2^2+y_1^2+y_2^2$$

$$=x_1^2+x_2^2-2x_1x_2+y_1^2+y_2^2-2y_1y_2$$

$$\leftrightarrow\quad 2x^2+2y^2-2xx_1-2xx_2-2yy_1-2yy_2+2x_1x_2+2y_1y_2=0$$

$$\leftrightarrow\quad x^2-xx_1-xx_2+x_1x_2+y^2-yy_1-yy_2+y_1y_2=0$$

$$\leftrightarrow\quad (x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

Example 7. Find the equation of the circle when the co-ordinates of the extremities of a diameter are $(-5,3)$ and $(+5,-2)$ Also find its radius and the coordinates of its centre.

Solution. The equation of a circle when the co-ordinates of extremities of a diameter are (x_1,y_1) and (x_2,y_2) is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

In this problem $x_1=-5,\ x_2=5$

$y_1=3,\ y_2=-2$

Hence the equation of the circle is

$$(x+5)(x-5)+(y-3)(y+2)=0$$

$$\leftrightarrow \quad x^2-25+y^2-y-6=0$$

$$\leftrightarrow \quad (x)^2+(y-\tfrac{1}{2})^2=31\tfrac{1}{4}$$

Hence the co-ordinates of its centre are $(0, \frac{1}{2})$ and its radius is $\sqrt{31\frac{1}{4}}$

We have already discussed that a line intersects a circle in two points if the measure of perpendicular segment from its centre to the line is less than the measure of its radius ; it intersects the circle in one point if the measure of perpendicular segment from its centre to the line equals the measure of its radius and the line does not intersect the circle at all if the length of the perpendicular segment from the centre of the circle to the line is greater than the measure of its radius. Let us prove the same by the help of analytical methods.

To prove it we need to make use of the following property of quadratic equations :—

A quadratic equation $ax^2+bx+c=0$ will have two roots (two values of x) if its discriminant b^2-4ac is positive ; it will have one value of x if its discriminant b^2-4ac equals zero and the solution of the equation will be an empty set if b^2-4ac is negative. Of course the cases $a=b=0$, $c\neq 0$ and $a=b=c=0$ are excluded as these do not make any sense from our present point of view.

Let the equation of the circle be $x^2+y^2=r^2$ with P(0,0) as its centre and with r as its radius. Let $y=mx+c$ be the equation of the line. Now to get the co-ordinates of points of intersection of the line and the circle we have to solve these equations simultaneously.

Now $x^2+y^2=r^2$ and $y=mx+c$.

$$\leftrightarrow \quad x^2+(mx+c)^2=r^2$$

$$\leftrightarrow \quad x^2+m^2x^2+c^2+2mcx=r^2$$

$$\leftrightarrow \quad x^2(1+m^2)+2mcx+c^2-r^2=0$$

Now $x^2(1+m^2)+2mcx+c^2-r^2=0$ is a quadratic equation and its discriminant is

$$(2mc)^2-4(1+m^2)(c^2-r^2)$$

$$=4m^2c^2-4c^2+4r^2-4m^2c^2+4m^2r^2$$

$$=4m^2r^2-4c^2+4r^2$$

Now the equation will have two roots, one root or no roots at all according to if $4m^2r^2-4c^2+4r^2$ is greater than, equal to or less than zero.

Accordingly if $r^2(1+m^2)-c^2 > = \text{or} < 0$

if $$r^2(1+m^2) > = \text{or} < c^2$$

or according as if $$r^2 > = \text{or} < \frac{c^2}{1+m^2}$$

or according as i $$r > = \text{or} < \frac{c}{\pm\sqrt{1+m^2}}$$

or according as if $\dfrac{c}{\pm\sqrt{1+m^2}} <, = \text{or} > r$

Now the length of perpendicular segment from P(0,0) to

$y=mx+c$ or $y-mx-c=0$ is $\dfrac{-c}{\pm\sqrt{1+m^2}}$

or it equals $\dfrac{c}{\sqrt{1+m^2}}$ in magnitude

Hence we prove that intersection of a line with a circle will consist of two points, one point or an empty set according as if the length of perpendicular segment from the centre of the circle to the line is less than, equal to or greater than the measure of the perpendicular segment. Now if the line intersects a circle in only one point, then it is a tangent to the circle. Now $y=mx+c$ will be a tangent to the circle if $c=r\sqrt{1+m^2}$

Hence $y=mx\pm r\sqrt{1+m^2}$ will be tangent to $x^2+y^2=r^2$. Now these are two equations. Hence we have two tangents to a circle from a point exterior to it.

The tangent to a circle at a point on it : Let $x^2+y^2=r^2$ be the given circle. Let (x', y') be a point on this circle. Now how to find the tangent to this circle at (x', y') ? Let $y-y'=m(x-x')$ be any line through the point (x',y') with a slope m. (Point—slope form of the equation)

Now $x^2+y^2=r^2$ and $y-y'=m(x-x')$

$$\leftrightarrow \quad x^2+[y'+m(x-x')]^2=r^2$$

$$\leftrightarrow \quad x^2+y'^2+m^2(x-x')^2+2y'm(x-x')=r^2$$

$$\leftrightarrow \quad x^2+m^2x^2+2y'mx-2m^2xx'+y'^2+m^2x'^2-2y'mx'=r^2$$

$$\leftrightarrow \quad x^2(1+m^2)+x(2my'-2m^2x')+y'^2+m^2x'^2-2mx'y'-r^2=0$$

Now the discriminant of this will be equal to zero if

$$(2my'-2m^2x')^2-4(1+m^2)(y'^2+m^2x'^2-2mx'y'-r^2)=0$$

or $$4m^2(r^2-x'^2)+8mx'y'+4r^2-4y'^2=0 \qquad ...(1)$$

Now as $(x'y')$ lies on the circle $x^2+y^2=r^2$

hence $x'^2+y'^2=r^2$; therefore $y'^2=r^2-x'^2$

and $\quad x'^2=r^2-y'^2$

Equation (1) becomes

$$4m^2y'^2+8mx'y'+4x'^2=0$$

or $$m^2y'^2+2mx'y'+x'^2=0$$

or $$(my'+x')^2=0$$

or $\quad my'+x'=0$ or $m=\dfrac{-x'}{y'}$ or the slope of the line should be $\dfrac{-x'}{y'}$. Hence the equation of the tangent line becomes

$$y-y'=\frac{-x'}{y'}(x-x')$$

$$\leftrightarrow \; yy'-y'^2=-xx'+x'^2$$

$$\leftrightarrow \; yy'+xx'=y'^2+x'^2$$

$$\leftrightarrow \; yy'+xx'=r^2 \qquad (\text{as } x'^2+y'^2=r^2)$$

Alternatively :

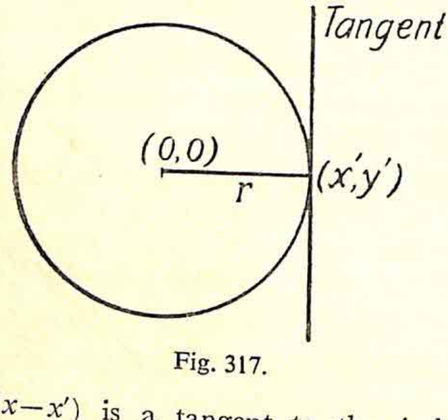


Fig. 317.

If $y-y'=m(x-x')$ is a tangent to the circle $x^2+y^2=r^2$ then the length of perpendicular segment from the origin to the given line will be equal to the measure of the radius.

Now the length of perpendicular segment from (0,0) to $y-y'=m(x-x')$ or $y-mx+(mx'-y')=0$ would be

$$\frac{0-0+(mx'-y')}{\sqrt{1+(-m)^2}}=\frac{mx'-y'}{\sqrt{1+m^2}}=r.$$

or $$m^2x'^2+y'^2-2mx'y'=r^2(1+m^2)$$

or $\quad m^2x'^2-2mx'y'+y'^2-r^2(1+m^2)=0$

or $\quad m^2x'^2-2mx'y'+y'^2-(x'^2+y'^2)(1+m^2)=0$

or $\quad m^2x'^2-2mx'y'+y'^2-x'^2-y'^2-m^2x'^2-m^2y'^2=0$

or $\quad -m^2y'^2-2mx'y'-x'^2=0$

or $\quad m^2y'^2+2mx'y'+x'^2=0$

and we once again arrive at the same equation.

Alternatively :

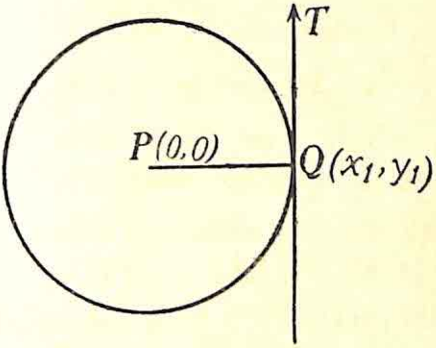


Fig. 318.

If $\overleftrightarrow{PQ}$ is the line perpendicular to the given line then the slope of $\overleftrightarrow{PQ}$ is $\dfrac{y_1-0}{x_1-0}=\dfrac{y_1}{x_1}$

Hence the slope of $\overleftrightarrow{QT}$ which is tangent to the circle and which will be perpendicular to $\overleftrightarrow{PQ}$ would be $\dfrac{-x_1}{y_1}$

Hence the equation

$$y-y_1=m(x-x_1) \text{ becomes}$$

$$y-y_1=\frac{-x_1}{y_1}\ (x-x_1)$$

$$\leftrightarrow \quad yy_1-y_1^2=-xx_1+x_1^2$$

$$\leftrightarrow \quad yy_1+xx_1=y_1^2+x_1^2$$

$$\leftrightarrow \quad yy_1+xx_1=r^2$$

Example. Find the equation of the tangent to the circle $x^2+y^2=41$ at the point Q(4, 5)

Solution : Equation of a tangent is given by the formula $xx'+yy'=r^2$ where (x',y') are the co-ordinates of Q.

In this problem $x'=4$ and $y'=5$ and $r^2=41$. Hence the equation of the tangent is

$$4x+5y=41$$

Problem Set

A

1. Find the equation of the circle and also draw their graph if the centre P and radius r are as follows ;

(*a*) P is at origin and $r=6$ units

(*b*) P $(-2, -3)$ and $r=7$ units

(*c*) P $(-3, 0)$ and $r=4$ units

(*d*) P $(0, 2)$ and $r=3$ units

(*e*) P $(0, 0)$ and $r=1$ units

(*f*) P $(2, 7)$ and $r=2$ units

2. Find the equation of a circle and also their graph if their centre is P and if they contain a point Q and if the co-ordinates of P and Q are as follows :

(*a*) P $(0, 0)$ and Q $(-3, -2)$

(*b*) P $(-3, 2)$ and Q $(7, -5)$

(*c*) P $(7, 0)$ and Q $(1, -3)$

(*d*) P $(-3, -5)$ and Q $(0, 0)$.

B

1. Find the co-ordinates of the centre and measure of the radius of the following circles. Also draw their graph.

(*a*) $x^2+y^2-4x+6y=0$

(*b*) $x^2+y^2-2x+4y+21=0$

(*c*) $x^2+y^2+5x-3y-11=0$

(*d*) $16+6x-x^2-y^2=0$

(*e*) $4x^2+4y^2-12x+4y-15=0$

(*f*) $2x^2+2y^2+5x-7y-10=0$

(*g*) $3x^2+3y^2+ax=0$

(*h*) $24-8x-6y-x^2-y^2=0.$

2. Given a point Q (x, y) and the sum of squares of Q from A $(-3, 1)$ and B $(0, 5)$ is equal to 5. Will the locus of Q be a circle ? If so, find the co-ordinates of its centre and the length of its radius.

3. Find the equation of each circle, the end points A, B of whose diameter are as under. Draw their graph and also find the co-ordinates of their centre and measure of their radius.

(*a*) A $(-2, -1)$; B $(3, 4)$
(*b*) A $(-1, 2)$; B $(4, -1)$
(*c*) A $(3, -9)$; B $(-5, 6)$
(*d*) A $(0, 0)$; B (a, b).

4. Find the equations of circles, the end points of whose diameters are the centres of the following circles.

(*a*) $x^2+y^2+2x+8y-1=0$; $x^2+y^2+x-3y-7=0$
(*b*) $2x^2+2y^2+5x-7y-10=0$;
$16x^2+16y^2-16x-8y-11=0$

5. Find the equations of circles containing the points :

(*a*) $(0, 0)$; $(5, 0)$ and $(0, 5)$
(*b*) $(0, 0)$; $(4, 5)$ and $(-4, 5)$
(*c*) $(a, 0)$; $(-a, 0)$ and $(0, b)$
(*d*) $(5, 7)$; $(8, 1)$ and $(1, 3)$
(*e*) $(1, 1)$; $(2, -1)$ and $(3, -2)$

6. Find the equations of circle which contains the origin and which cuts off intercepts C_1 and C_2 from the axes of x and y.

7. The equations of sides of a rectangle ABCD are as follows :

$$x+3y-7=0 \qquad x+3y-12=0 ; \qquad 3x-y+9=0$$

and

$$3x-y-6=0.$$

Find the equations of the circles in each of the following cases :

(1) A circle containing A, B and C.
(2) A circle containing A, B and D.
(3) A circle containing B, C and D.
(4) A circle on $\overline{AC}$ as diameter.
(5) A circle on $\overline{BD}$ as diameter.

8. Find the equation of the circle which passes through the origin, whose centre lies on the line $x-y=0$ and whose radius is 5.

Solution. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0$$

Now as the circle contains the origin, hence

$$0+0+0+0+c=0 \qquad \text{Hence} \qquad c=0$$

The equation of the circle becomes

$$x^2+y^2+2gx+2fy=0$$

Now find the coordinates of its centre and the measure of its radius, we write it as

$$(x^2+2gx+g^2)+(y^2+2fy+f^2)=g^2+f^2$$

$$\leftrightarrow \quad (x+g)^2+(y+f)^2=(\sqrt{g^2+f^2})^2$$

Hence the coordinates of the centre are $(-g, -f)$ and the measure of its radius is $\sqrt{g^2+f^2}$.

Therefore $\sqrt{g^2+f^2}=5$ or $g^2+f^2=25$ ...(1)

Also P $(-g, -f)$ lies or $x-y=0$

Hence $-g+f=0$ or $g=f$ ...(2)

Now $g^2+f^2=25$ and $g=f$

$\leftrightarrow \quad f^2+f^2=25 \leftrightarrow 2f^2=25$

$\leftrightarrow \quad f^2=\dfrac{25}{2} \quad \leftrightarrow \quad f=\dfrac{5}{\sqrt{2}}$

Also then $g=\dfrac{5}{\sqrt{2}}$

Hence the equation of the circle becomes

$$x^2+y^2+\frac{10}{\sqrt{2}}x+\frac{10}{\sqrt{2}}y=0.$$

9. Find the equation of the circle which contains the point (3, 4) ; whose centre lies on $3y-4x=0$ and whose radius is $2\frac{1}{2}$ units.

10. Find the equation of the circle containing the points $(1, -2)$ and $(4, -3)$ and whose centre is on the line $3x+4y-7=0$.

C

1. Find the equation of the tangent to the following circles :

(a) $x^2+y^2=149$ at the point (10, 7)
(b) $x^2+y^2=25$ at the point (3, 4)
(c) $2x^2+2y^2=4$ at the point (1, 1)
(d) $x^2+y^2=7$ at the the point (3, 2).

2. Find the equation of the circle whose centre is at (2, 1) and which is tangent to x-axis.

3. Find the equation of the circle whose radius is 4 and which is tangent to both axes.

4. Find the equation of the circle whose radius is 5 and which is concentric to $x^2+y^2+2x+4y+4=0$.

5. Find the equation of a circle which is tangent to both axes and whose centre lies on $y=x$.

6. Find the equation of the tangent with centre at $(2, -3)$ and radius 5 units.

7. Find the locus of P such that the sum of its square from $(-2, -3)$ and $(4, -5)$ is 23.

CHAPTER XVI

OTHER NEW GEOMETRIES

(*Euclid displaced*)

We have developed Euclidean geometry as a logical deductive system little caring about its applicability to the real life situations. Of course, we adopted those axioms which are true within the limits of experiment on the surface of a sheet of paper. But is the plane of sheet really plane ? Is it not covered with small ridges and furrows ? What will our sheet look like under a powerful microscope ? Will our axioms and postulates hold if the sheet has these ridges ? Is every small portion of our space as 'plane' as that of a sheet of paper ?

In fact our axioms and postulates are more likely not to hold if we consider their applicability to the space in which we live or even to the surface of our planet which has lots of hills, ridges and furrows. A mathematician, however, is not interested in applicability of the system that he develops. He does not care whether his assumptions hold in real life or not. He only cares to see that his assumptions are consistent which means that none of his assumptions should contradict another one. If his set of assumptions is consistent then there will be no contradictions in the theorems that he proves with the help of these assumptions and the theorems which have not been discovered or proved but which could be proved within his system. Suppose a theorem can be proved within the system whose proof is flawless but which contradicts the result of another theorem of the system then it implies that the set of assumptions of the system are not consistent and the mathematician need to examine his set of assumptions in order to weed out or modify the one that has created the trouble.

To illustrate this point, let us discuss the following set of axioms.

A_1 : The sum of two numbers is another number of the system.

A_2 : Every number has a unique immediate successor.

A_3 : Every number is a unique immediate successor of one and only one number.

A_4 : Given any two numbers, then either the first number is smaller than the second or the first number equals the second or the first number is larger than the second.

A_5 : The largest number is 12.

Now it is evident that the first four Axioms are true for the set of natural numbers while the fifth is not. 12 is an immediate successor of 11 but 12 does not have an immediate successor as according to A_5, there is no number like 13 existing in the system. Also $7+5=12$ but what about $8+5$? So set of natural numbers cannot be a model for this set of axioms or in other words we cannot establish the consistency of this set of axioms if we take the set of natural numbers as a model. Let us look for some other model. It will be useful to remember that to establish the consistency of a set of axioms it is enough to find one model for which there are no contradictions in the set of axioms. Let us consider what is known as 'clock arithmetic'. In this we have numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. In this the sum of two numbers is another number. $2+9=11$; $3+9=12$; $4+9=1$ are its examples. If we start from the step 4 on the clock and count 9 steps further, we arrive at 1. Also $8+5=1$. $3-8=7$ in this system which of course we can verify. So in this system A_1 holds. A_2 is also satisfied as every number has as immediate successor. The immediate successor of 12 is 1. Also A_3 holds. A_5 is also satisfied.

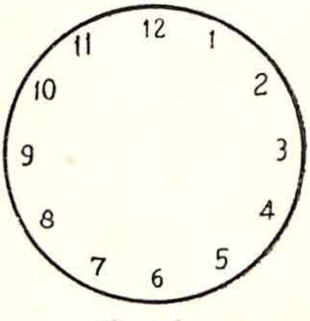


Fig. 319.

But what about A_4 ? Given any two numbers in the system then one of these is less than or equal to or greater than the other ?

Let us consider 7 and 5. $7-2=5$. Hence 7 is larger than 5. But also $7+10=5$ (How ?). Hence 7 is smaller than 5. So A_4 does not hold. In actual life we do not know whether 7 o' clock comes after 5 o' clock or before it ? (How is it ?) So we have not been able to establish the consistency of this set of axioms. May be the reader could do so by finding a suitable model ! ! !

By taking suitable sets of axioms, the mathematicians have developed new geometries—known as finite projective geometries. Let us discuss one as an example.

Let the set of axioms be as follows :

A_1 : If P_1 and P_2 are any two points, there is atleast one line which contains both P_1 and P_2.

A_2 : If P_1 and P_2 are any two points, there is atmost one line that contains both P_1 and P_2.

A_3 : If l_1 and l_2 are any two lines, there is atleast one point that lies on both l_1 and l_2.

A_4 : Every line contains atleast three points.

A_5 : If l is any line, there is atleast one point which does not lie on l.

A_6 : There exists atleast one line.

Now if we look at this set of axioms closely then we find that A_3 states that every two lines must intersect. In other words it excludes the existence of parallel lines. It contradicts the Euclidean parallel postulate. Of course it is easy to establish that this set of axioms is consistent.

Let us consider the figure 320 as a model, A_1 states that if P_1 and P_2 are any two points then there is atleast one line that contains P_1 and P_2. Now this is satisfied if we consider P_1 and P_2 or P_2 and P_3 etc. as our points. But is it satisfied if we consider P_4 and P_6 ? Is there any line that contains both P_4 and P_6 ? What about the arc of the inscribed circle joining P_4 and P_6 ? Cannot it be our line ? Yes, of course. In this system, we have not defined our lines as ones which are only 'straight'. Simply our line has to be a set of points. Hence A_1 is satisfied. A_2 is also satisfied since there is only one line containing any two given points A_3 is also satisfied as every two lines have one point in common. It may be noted that we are taking the inscribed circle as a set or a line. A_4 is also satisfied as every line contains atleast three points. Similarly A_5 and A_6 are satisfied. Hence our set of axioms is consistent. It may be noted here that a 'point' or a 'line' should not mean the 'point' or the 'line' as we mean them in Euclidean geometry. A point could be any object that has the properties that the set of axioms says it should have. Similarly a line could also be any object. It may, however be noted that this axiomatic system consists of two undefined terms—points and lines ; one undefined relation—contains or lies and six assumptions. Now let us prove some theorems in this very system.

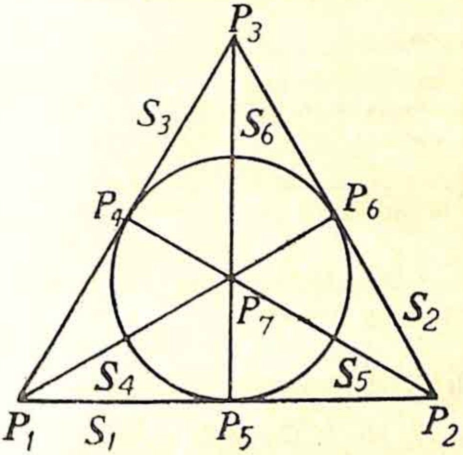


Fig. 320.

Theorem 1. There exists atleast one point.

Statement	*Reason*
(1) There exists a line l	(1) A_6
(2) If l is a line then there is atleast one point, P, that does not lie on l	(2) A_5
(3) A point P exists	(3) Statement 2.

Instead of making use of A_5, we could have made use of A_4 and arrived at the required result. (How ?)

Theorem 2. Two points determine exactly one line.

Statement	*Reason*
(1) There exists atleast one line containing both P_1 and P_2	(1) A_1
(2) There exists atmost one line containing both P_1 and P_2	(2) A_2
(3) There exists exactly one line containing both P_1 and P_2	(5) Statements 2 and 3.

Theorem 3. Two lines have atmost one point in common.

Proof. Let us assume the contrary and state that two lines l_1 and l_2 have two points P_1 and P_2 in common. So there are two lines l_1 and l_2 containing both P_1 and P_2. But this contradicts A_2. Hence the assumption that l_1 and l_2 have two points in common has to be abandoned.

Theorem 4. Two lines have exactly one point in common. Theorem 3 & A_3 helps us in proving this. (How ?)

Theorem 5. Given a point P. There is atleast one line that does not pass through P.

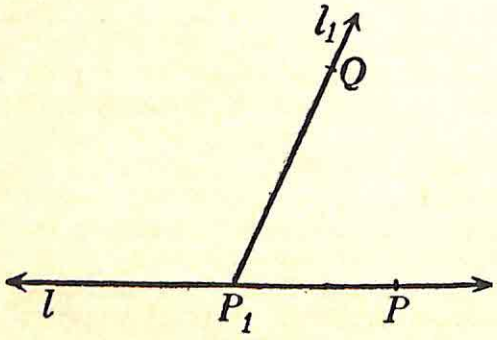


Fig. 321.

Statement	*Reason*
(1) There exists a line l	(1) A_6

If l does not pass through P, then we have nothing further to prove. Let us assume that l passes through P.

Statement	*Reason*
(2) There exists a point Q that does not lie on l	(2) A_5
(3) l has a point $P' \neq P$	(3) A_4
(4) There is one and only one line l' containing Q and P_1	(4) Theorem 2
(5) $l \neq l'$	(5) Since l' contains Q while l does not
(6) P will not lie on l'	(6) Since l and l' can have only one point in common and l and l' have a point P' in common and $P' \neq P$
(7) l' is a line that does not contain P.	(7) Statement 6.

Theorem 6. Every point lies on atleast three lines.

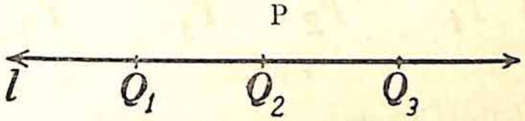


Fig. 321.

Statement	*Reasan*
(1) There is a line l	(1) A_6
(2) There is a point P that does not lie on l	(2) A_5
(3) l contains points Q_1, Q_2, Q_3 such that $Q_1 \neq Q_2 \neq Q_3$	(3) A_4
(4) P, Q_1 determine a line l_1 ; P_1, Q_2 determine a line l_2 ; P, Q_3 determine a line l_3	(4) A_1
(5) $l_1 \neq l_2$; $l_2 \neq l_3$; $l_1 \neq l_3$	(5) If $l_1 = l_2 = l_3$ then l_1, l_2, l_3 will have atleast three points Q_1, Q_2, Q_3 in common with l which will contradict theorem 3
(6) P lies on l_1, l_2 and l_3	(6) Statements 4 and 5.

Theorem 7. If there exists one line that contains exactly n points, then every line contains exactly n points.

Statement	*Reason*
(1) There is a line l containing $P_1, P_2, P_3, \ldots P_n$	(1) Given
(2) There is a point Q not lying on l	(2) A_5
(3) $(Q, P_1) : (Q, P_2) ; (Q, P_3) \ldots (Q, P_n)$ determine lines $l_1, l_2, l_3 \ldots l_n$	(3) A_1
(4) On l_2 there is a point P'_2	(4) A_4
(5) (P_1, P_2') determine a line l'	(5) A_1
(6) (l', l_1) ; $(l', l_2), (l', l_3) \ldots (l', l_n)$ detetermine points $P'_1, P'_2, P'_3 \ldots P'_n$ $(P_1 = P'_1)$ which are distinct from each other	(6) Theorem 4
(7) l' contains atleast n points	(7) Statement 6.

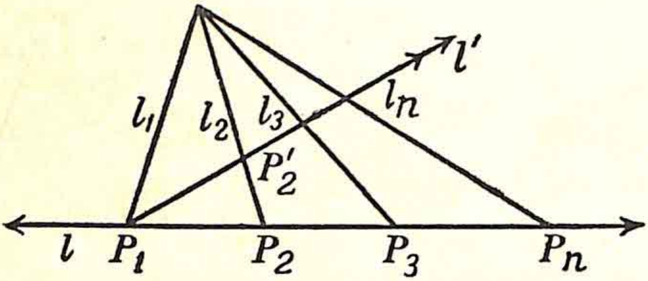


Fig. 322.

Now we must show that l' contains exactly n points. Let us assume the contrary and state that l' contains a point P'_{n+1}.

Now (Q, P'_{n+1}) determine a line l_{n+1}. (l_{n+1}, l) will have a point P_{n+1} in common (Theorem 4). It implies that l contains $n+1$ points which contradicts the hypothesis.

Hence the assumption that l' contains P_{n+1} has to be abandoned.

Theorem 8. If there exists one line which contains exactly n points, then exactly n lines pass through every point.

Statement	*Reason*
(1) There is a line l.	(1) A_6

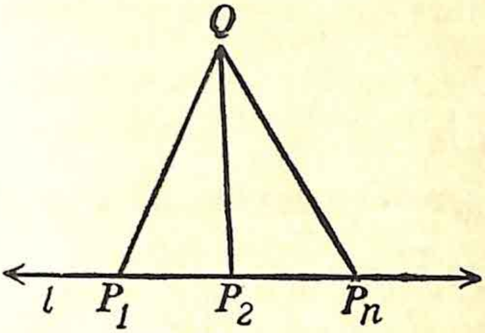


Fig. 323.

(2) There is a point Q that does not lie on l.	(2) A_5
(3) l contains exactly n points $P_1, P_2, P_3 \ldots P_n$	(3) Given
(4) $(Q, P_1), (Q, P_2)\ (Q, P_3)\ldots(Q, P_n)$ determine lines $l_1, l_2, l_3, \ldots l_n$ such that $l_1 \neq l_2 \neq l_3 \ldots \neq l_4$	(4) Theorem 2.
(5) At least n lines $l_1, l_2, l_3, \ldots l_n$ pass through Q.	(5) Statement 4

Now we have to prove that atmost n lines pass through Q. Let us assume the contrary and state that an additional line l_{n+1} passes through Q. Now (l_{n+1}, l) will have a point P_{n+1} in common (Theorem 4) Also $P_{n+1} \neq P_1 \neq P_2, \ldots P_n$. It implies that l contains $n+1$ points $P_1, P_2, P_3 \ldots P_n, P_{n+1}$ which contradicts the hypothesis. Hence the assumption that an additional line l_{n+1} passes through Q has to be abandoned,

Theorem 9. If there exists one line which contains exactly n points, then the system contains exactly n^2-n+1 points.

Statement	*Reason*
(1) Through point Q, n lines $l_1, l_2, l_3 \ldots l_n$ pass	(1) Theorem 8.
(2) Every one of $l_1, l_2, l_3 \ldots l_n$ contains $n-1$ points besides Q.	(2) Theorem 7.
(3) n lines contain $n(n-1)$ points besides Q.	(3) Statement 3.
(4) The sytem contains $n(n-1)+1$ or n^2-n+1 points	(4) Statement 3.

Theorem 10. If there exists one line which contains n points, then the system contains exactly n^2-n+1 lines. (Refer Fig. 324).

Statement	*Reason*
(1) l contains n points $P_1, P_2, P_3, \ldots P_n$	(1) Given
(2) Through each one of n points $P_1, P_2, P_3, \ldots P_n$, $(n-1)$ lines pass besides l.	(2) Theorem 8.
(3) The system contains $n(n-1)$ lines besides l.	(3) Statement 2.
(4) The system contains $n(n-1)+1$ or n^2-n+1 lines.	(4) Statement 3.

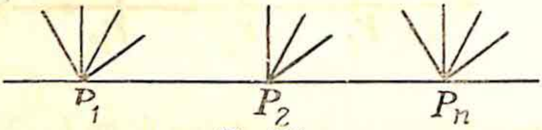


Fig: 324.

Let us consider another finite geometry. We have two undefined elements, points & lines, one undefined relation 'on'. Of course 'point on a line' or 'a line on a point' shall be understood to have same meaning. We take the following axioms.

A_1 :—Every pair of lines is on at least one point.

A_2 :—Each pair of lines is on not more than one point.

A_3 :—Each point is on atleast two lines.

A_4 :—Each point is on not more than two lines.

A_5 :—The total number of lines is 4.

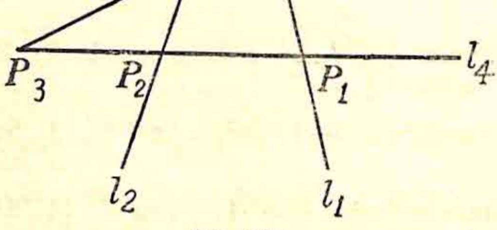


Fig. 325.

Now A_1 and A_2 state that two lines meet at one and only one point. A_3 and A_4 state that each point is on exactly two lines. The geometric figure given here would serve as a concrete model of this system. All the axioms of the system are satisfied. Hence we establish that the set of assumptions is consistent. Following theorems can be easily proved in this system.

Theorem 1. Not all lines pass through the same point.

Theorem 2. Two distinct lines have exactly one point in common.

Theorem 3. Exactly two lines pass through each point.

Theorem 4. Every line contains exactly three points.

Theorem 5. The system contains six points in all.

We can construct other models by using (1) Colours and colour schemes (2) containers and objects to be put in it (3) Teachers and committees etc. as our undefined terms instead of 'points and lines'. Let our 'points' be the Maths. teachers in a school and let the 'lines' be the 'committees' formed to discuss curriculum, methods of teaching, club activities etc. etc. Let the set of teachers be :—

(1) Sengupta (2) Kapur (3) Chawla (4) Trivedi (5) Aggarwal (6) Sharma

Let the committees be :

(1) Curriculum committees. (Sengupta, Kapur, Chawla)

(2) Methodology Committee. (Sengupta, Aggarwal, Sharma)

(3) Maths. Club Committee. (Kapur, Trivedi, Sharma)

(4) A.V. aids Committee. (Chawla, Trivedi, Aggarwal)

Now our assumptions will be interpreted as follows :

(*a*) Each pair of committees has atleast one teacher in common.

(*b*) Each pair of committees has atmost one teacher in common.

(*c*) Each teacher is atleast on two committees.

(*d*) Each teacher is on not more than two committees.

(*e*) There are four committees in all.

The theorems will have to be re-worded as follows :

Theorem 1. Not all the committees have the same teacher.

Theorem 2. Two different committees have exactly one teacher in common.

Theorem 3. Exactly two Committees have a teacher in common.

Theorem 4. Every Committee has exactly three teachers.

Theorem 5. There are six teachers in all.

So we could find two models for the same set of assumptions. The reader can formulate as many more models as he wants to. Also the reader can develop his own finite geometry by taking a suitable set of assumptions involving undefined terms and relations which should be consistent. Then he can build up his own body of theorems.

Two other geometries which are of recent origin and on which we want to just touch upon are (1) Lobatchevskian or Hyperbolic geometry (2) Riemanian geometry. For more than two thousand years, the system formulated by Euclid (Euclidean geometry) occupied a position of authority. The rules formulated by Euclid for geometric relations were assumed to be true for the space. It was thought that space obeys Euclid and Euclid obeys space. In other words, it was assumed that space fulfills a certain code of rules, axioms, definition, postulates all laid down in one form or the other in the Euclid's Elements of geometry. We can derive following postulates about space from postulates of Euclid.

(1) Space is a continuous aggregate of points and not a discrete aggregate. This we derive from the following postulates of Euclid :—

(a) A straight line can be drawn from any point to any point.
(b) A finite straight line can be produced indefinitely in either direction.

Now if the space consists of discrete aggregate of points then it will not be possible to have (a) as well as (b) Moreover Euclid proved the proposition that 'Every finite straight line may be bisected'. Now if space consists of an odd number of points then it may not be possible to bisect it as there may not be a point at a particular position at which the given straight line is divided into congruent lines. So Euclid must have assumed this postulate of continuity which may be taken to mean that between every two points on a line, there is an infinite number of points.

(2) The next postulate concerning space that Euclid assumed could be described as that of elementary flatness as while finding the circumference of a circle we assumed that the length of an infinitely small arc of a circle of a large radius equals the length of the chord joining the ends of the arc. It implies that to an observer perceiving a portion of the surface of a very large sphere such as earth, the portion of the surface will appear to be flat.

(3) Another postulate about space that Euclid seems to have assumed is the postulate of superposition. According to this, a body can be moved about in space without altering its shape or size. It implies that during motion, all the lines and angles of the 'body" remain unaltered. Also if a figure has been drawn on some surface then it can be moved about without alteration in its size or shape. This, of course, will hold if the given surface is a part of

'perfectly smooth' sphere but it will not hold if the given surface is a part of surface, parts of which are 'more curved' than others.

(4) The fourth postulate that Euclid seems to have assumed is the postulate of similarity. According to this, any figure may be magnified or diminished to any 'degree' without altering its shape. In other words if one figure has been constructed in one part of the space, it may be reconstructed to any scale whatsoever in any other part of the space. It implies that lengths of all lines would be altered while none of the angles would be altered. Basically, this involves following two postulates of Euclid.

(i) Two straight lines cannot enclose a space.

(ii) Lines in one plane which never meet make equal angles with every other line. The second one, of course, is our familiar parallel postulate. To even earlier mathematicians, parallel postulate appeared to be 'much less obvious' as compared to Euclid's other postulates.

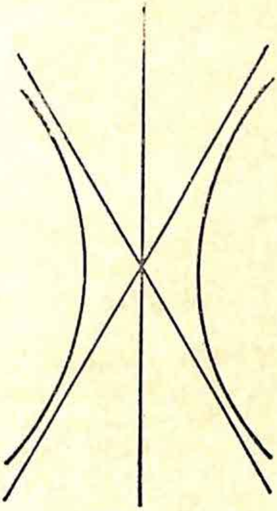


Fig. 326.

So its validity had always been doubted. The ancient Greeks knew about hyperbola and knew that although its branches approach the corresponding asymptotes, yet those never met them. They doubted that straight lines may be behaving like that. This doubt continued to exist in the minds of the mathematicians for over two thousand years. They tried to deduce the 'parallel postulate' as a theorem from other postulates of Euclid. Jesuit Gerolamo Saccheri (1667—1733) a professor at the University of Pavia, adopted a hypothesis 'contrary' to parallel postulate and explored its consequences. His aim was to vindicate Euclid by proving that one arrives at absurd results when one assumes a hypothesis contrary to parallel postulate. He derived a long chain of results consistent among themselves and without any contradiction but quite different from Euclidean geometry. Then he erroneously concluded that the hypothesis has to be abandoned on account of these results. It was left to Russian mathematician Nicholas Ivanovich Lobatchevski (1793—1856) and Hungrarian mathematician Johann Bolayai (1802—1860) to dovelop, independent of each other, a new geometry known as hyperbolic geometry by assuming a postulate quite different from Euclid's parallel postulate.

Lobatchevski assumed that 'two straight lines cannot enclose a space' is a valid statement. In other words he assumed that if two lines diverge then those go on diverging and will diverge for ever and will not converge. He, however, doubted the validity of Euclid's parallel postulate which as originally stated by validity.

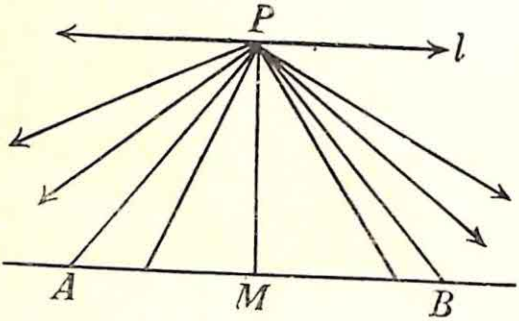


Fig. 327.

Euclid is as follows :—'If a straight line falling on two straight lines make interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely will meet on the side on which are the angles less than two right angles. We can put the parallel postulate in the following form also. If $\overleftrightarrow{AB}$ is any line and if P is any point outside the line and $\overline{PM}$ is the perpendicular segment from P to $\overleftrightarrow{AB}$. If through P, we draw lines in all directions then these lines will meet $\overleftrightarrow{AB}$, either on B-side of M or on A—side of M excepting one line *l* which even on being produced indefinitely will not meet $\overleftrightarrow{AB}$ at all. The line *l* will be said to be parallel to $\overleftrightarrow{AB}$.

The parallel postulate that Lobatchevski enunciated was as follows :—

If $\overleftrightarrow{AB}$ is any line and if P is any point outside it then P will be the end point of two non-collinear rays, $\overrightarrow{PM}$ and $\overrightarrow{PN}$ making a finite angle $\angle$MPN such that the rays $\overrightarrow{PM}$ and $\overrightarrow{PN}$ do not intersect the line $\overleftrightarrow{AB}$ and also every ray between $\overrightarrow{PM}$ and $\overrightarrow{PN}$ does not intersect $\overleftrightarrow{AB}$. The two distinct lines through P containing $\overrightarrow{PM}$ and $\overrightarrow{PN}$ would be the lines parallel to $\overleftrightarrow{AB}$. In other words Lobatchevski assumed the existence of atleast two lines through P both parallel to $\overleftrightarrow{AB}$. This position is similar to the case of a hyperbola and its assymptotes. If $\overleftrightarrow{AB}$ is a branch of a hyperbola then the lines through P are the assymptotes. If two trumpets are placed with their large ends together, the surface formed called a pseudosphere is the surface suitable for Lobachevskian geometry. Due to this difference about parallel postulate, hyperbolic geometry does not need postulates about plane. (No. 4, 5, 6, 7). Also rectangles do not exist in hyperbolic geometry, hence postulates about rectangles *viz.*, (Postulate No. 18, 19, 20, 21) do not have any place in hyperbolic geometry. With this change in postulates of Euclidean geometry an axiomatic system is developed and certain theorems are proved which are quite different from the analogous theorems of Euclidean geometry. Some of the typical examples are :—

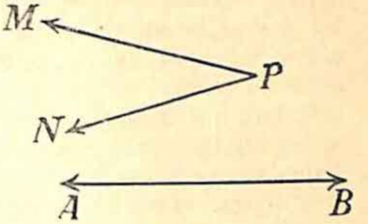


Fig. 328.

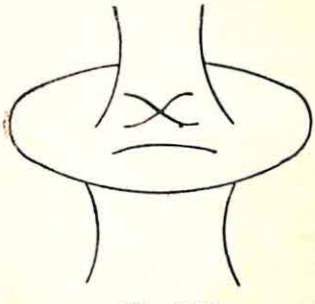


Fig. 329.

(1) No quadrilateral is a rectangle. If two angles of a quadrilateral are right angles then the other two are acute. If the three angles of a quadrilateral are right angles then the fourth is acute.

(2) The sum of the measures of any triangle (including a right triangle) is less than 180.

(3) If there exists a correspondence between two triangles in which the corresponding angles are congruent, then the triangles are congruent. No two triangles are ever similar except when those are also congruent.

Of course Saccheri did vindicate Euclid when he failed to prove the parallel postulate as a theorem from Euclid's other postulates. Its reason is that Saccheri's failure was due to the fact that Euclid's parallel postulate is independent of his other postulates. Also all the postulates of Euclid are consistent as none of theorems derived from them contradicts any other theorem. This only means that system developed by Euclid is valid and also the system developed by Lobatchevski is valid. A mathematician is neither worried about the practical applicability of any system nor about the quality of 'self-evidence' of any particular postulate. In his view the postulates are essentially arbitrary and need not be self-evident. In this sense Euclid was the most modern mathematician as he appears to have to very well understood this essential characteristic of the postulates. This whole discussion however leads us to state that our space may not be Euclidean as in space the sum of angles of every triangle may not be 180.

Yet another alternative to Euclid's parallel postulates was assumed by G.B.F. Riemann (1826—1866) and as a consequence of this, developed another geometry which is commonly known as elliptic geometry or Riemanian geometry. It is as under :—

The Riemanian parallel postulate :—No two lines in the same plane are ever parallel.

In other words, in this system, the existence of parallel lines is denied. If the surface taken is the surface of the sphere and if the great circles of spere be taken as lines then clearly parallel lines do not exist as all great circles intersect and that two in two points. Moreover the same two points may not determine only one line as the points N and S lie on infinite number of great circles. Also these 'lines' do not come to an end and yet these are of finite length. The concept of betweenness about which we have been talking in Euclidean geometry does not hold as if there are any three points on a great circle then it is not possible to state which one of these is between the other two. Also from any external point, there may exist one or many perpendiculars. From North pole or South pole infinitely many perpendiculars to the equator exist. The sum of the angles of a triangle is greater than two right angles. In some cases two angles of a triangle may each be a right angle. An example would be a triangle determined by any two points on the equator with either the North or the South pole. Of course, in such a case, then the measure of an exterior angle may be equal to the measure of an opposite interior angle. In this system, however, every line separates a plane into two half planes as is the case with the Euclidean geometry. Einstein theory of relativity takes the space to be curved like a sphere—finite but unbounded—a concept which is very difficult to imagine. However Riemanian geometry fits this idea and it has been used to develop the Einstein's theory of the universe. Whether the space is Euclidean or non Euclidean, however, is still an open question.

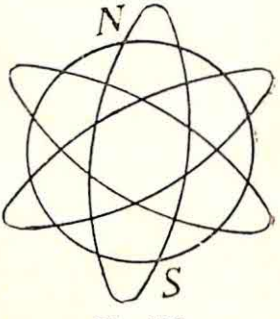


Fig. 330.

Of course, with all the efforts put in by Lobatchevski, Riemann, Saccheri and others, it could not be clearly demonstrated that it is impossible to prove Euclidean parallel postulate on the basis of his other postulates. It was an Italian mathematician Beltrami (1835—1900) who exhibited a particular model within Euclidean geometry in which the postulates of hyperbolic geometry were satisfied. Thus he proved that Euclid's parallel postulate is independent of others and finally settled the controversy. Also it was established that there exist (i) a mathematical model that satisfies the Euclidean postulates and (ii) a mathematical model that satisfies the Lobatchevskian postulates.

This discussion leads us to state that different kinds of 'lines' and 'surfaces' exist which are not necessarily 'straight' or 'plane' as we generally visualise them to be. Also we have different kinds of three-dimensional spaces based on these kinds of lines and surfaces. What kind of a space do we live in ? This is really a big question. We know that the axioms of Euclidean geometry are true within the limits of experiment on the surface of a sheet of paper ignoring, however, the ridges and furrows that it may have. The surface of small portions of space may be analogus to the sheet of paper (with little ridges and furrows of course) which, however on average is flat. But what about the whole space ? Is it Euclidean ? Is it Lobatchevskian ? Is it Riemanian ? Is it something different from all these ? May be some future mathematician will find an answer to this question.

In our earlier discussion we have stated that in the hyperbolic geometry, there are an infinite number of parallels while in the Riemann geometry which is also known as elliptic geometry, the parallels do not exist. We explained this characteristic of the elliptic geometry by taking the spherical geometry as a model. This model essentially consisted of the surface of a sphere and we viewed this surface as analogous to a plane in the Euclidean geometry. On our model, the surface of the sphere, the shortest distance between any two points was taken to be the portion of the great circle passing through these points which is also known as the geodesic which is analogous to a 'straight' line in a Euclidean plane. The equator and the meredians of the earth are the familiar examples of a geodesic. This model we took simply with a view to help us understand intutively the geometrical surface of a plane in Riemanian space or elliptic space or elliptic geometry. It is however stated that in spherical geometry, two great circles (or two lines on a sphere) always have two points in common, while the lines in a plane in the Riemanian space have only one point in common. Therefore we must be careful not to overextend the analogy between the surface of a sphere and the Riemannian plane. While making this analogy, we must assume that each point in the Riemanian plane corresponds to a pair of opposite points on the surface of a sphere. For example, starting from the North pole and travelling to the South pole on the earth would correspond to starting from one point on the Riemanian plane, travelling in a straight line on the plane and returning to the same point. All geodesic lines in Riemanian space have the same finite length and are closed like the circumference of a circle. This means extending our intution to far but this is how the facts stand. Also in Riemanian space, the ratio of circle's circumference to its diameter is always less than *Pi* (π). Suppose we draw circle on the surface of the earth with the North pole as centre, then its diameter would be the arc of a great circle containing the north and the South pole and which will be having its end points on the circumference of the circle we have drawn. Clearly the length of this arc will be greater than any straight line (in Eulidean sense) joining its points and passing through the interior of the sphere.

As the ratio of the circumference of the circle to 'this straight line' will be *Pi*, the ratio of the circumference of the circle to its diameter (the length of the arc) will be less than *Pi*. (It may be noted that the interior of the sphere does not belong to the surface of the sphere. Hence any line passing through the interior of the sphere cannot be the diameter of the circle that we have drawn on the surface of the sphere. (The diameter must wholly lie in the surface on which the circle has been drawn). This example of drawing a circle on the surface of a sphere is analogous to a circle drawn on a Riemanian plane.

In a Lobachevski space it is just the other way. The ratio of the circumference of a circle drawn on an hyperbolic plane to its diameter is greater than *Pi*. Another difference between a Euclidean plane, a Riemanian plane and a Lobachevskian plane is regarding the measure of the curvature of the surface of the plane at any given point. While the measure of curvature of a Euclidean plane is zero, it is less than zero in the case of Lobachevskian plane. The measure of curvature of a Lobachevskian plane is negative and constant. There is an infinite number of Lobachevskian geometries and each is characterised by the value of the curvature that a plane in that geometry possesses. In other words for any Lobachevski space, there is a certain fixed negative number which is the measure of curvature of any point in any plane in that space. The measure of curvature of any plane in a Riemanian geometry is greater than zero and we can characterise a Riemanian geometry on the similar lines as we adopted for a Lohachevskian geometry. We can do so by saying that there is a certain positive number which is the measure of curvature for any point on any plane in that space.

In other words, the measure of curvature at any point in any plane in such a space is the same.

How do we measure the curvature of a surface at a given point ? In Euclidean geometry, we proceed as follows. Let l be a line. Let l_1 be an infinitesimal part of it. Then the radius of curvature of this line would be the radius of the circle which, so to speak, will coincide with that infinitesimal part of the line. If a line is 'straight' then the radius of curvature is very long. If, however, it is strongly curved then it is short. The measure of curvature of the line is the reciprocal of the radius of curvature. Curvature however is a technical term and is not to be understood in an ordinary sense. To measure the curvature of surface of a Lobachevski plane or a Riemanian plane, at any point, we take two geodesics that intersect at that point and extend in two directions which are known as the principal directions of the surface at that point. One of these directions will give the maximum curvature at that point while the other will give the minimum curvature. The curvature of the surface at that point is defined as the product of the reciprocals of the two radii of curvature of the two geodesics.

It should however be clearly understood that measure of curvature as applied to the non-Euclidean planes does not mean that these planes "curve" in the ordinary sense of the term. When Einestein made use of the non-Euclidean geometries in his general theory of Relativity, certain authors coined the terms like 'curved planes' and 'curved space' which were incorrect. Instead they should have stated that there is a number R which gives the measure of curvature of the surface which is positive inside the sun and which is negative in the suns gravitational field and which approaches to zero when we go farther away from the sun.

The following table summarises the main points of difference between various geometries that we have discussed here.

Geometry	*Number of Parallels*	*Sum of angles of a triangle*	*Ratio of circumference to diameter of a circle*	*Measure of curvature*
Lobachevski	∞	<180	$> \pi$	<0
Euclid	1	180	π	0
Riemann	0	>180	$< \pi$	>0

Problem Set

A mathematical system consists of the following postulates.

P_1 : If P′ Q are any two points then there exists one and only one line that contains them both.

P_2 ; Every line is a set of points and contains atleast two points.

P_3 : If there is a line l and a point P which does not lie on l, then there is one and only one line through P parallel to l.

P_4 : Space contains atleast three non-collinear points.

Construct a geometric figure that may serve as a model of this system. Also prove the following theorems :

If there exists one line containg n points then :

(*a*) Every line contains exactly n points.

(*b*) Every point lies on exactly $(n+1)$ lines.

(*c*) Space contains exactly n^2 points.

(*d*) Space contains exactly $n\ (n+1)$ lines.

2. Given a mathematical system consisting of following postulates :

P_1 : If P, Q are any two points, there is exactly one line that contains them both.

P_2 : If P, Q, R are any three non collinear points, then there is exactly one plane that contains them.

P_2 : If two planes intersect, then their intersection is a line.

P_4 : If two points, P, Q lie in a plane, then the line containing P, Q lies in that plane.

P_5 : Every line contains atleast two points, every plane contains atleast three non-collinear points and space contains atleast four non-collinear non-coplanar points.

(There are what are commonly known as Incidence postulates.)

Construct a three dimensional geometrical figure which may serve as a model for this system. Can a prism serve the purpose ? Also prove the following theorems :

(1) Two distinct lines intersect in atmost one point.

(2) A line and a point not lying on it, determine one and only one plane.

(3) If two lines have a common point, then their union determines one and only plane.

3. A matematical system consists of following postulates.

P_1 : A line contains exactly two points.

P_2 : Exactly four lines pass through every point.

P_3 : Two points determine exactly one line.

P_4 : There exists atleast one point.

Draw a geometrical figure satisfying these postulates. Also prove the following theorems :

(1) There are exactly five points.

(2) There are exactly ten lines.

Rubber Sheet Geometry: (*Topology*).

In Euclidean geometry we have made use of relations like $=, \sim, \cong, >, <$ etc. The proof of every theorem involved the use of one or more of these relations. Can we have a geometry in which we may not have to use any of these relations ?

Suppose we have a triangle ABC made of an elastic band. Then can we deform this triangle into any one of the following diagrams ?

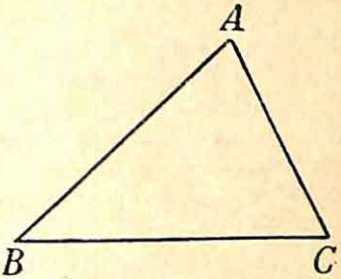
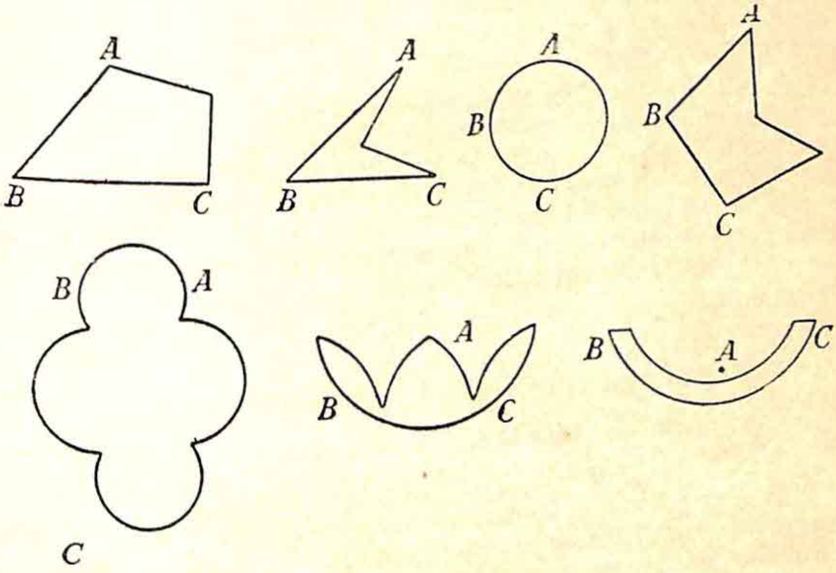


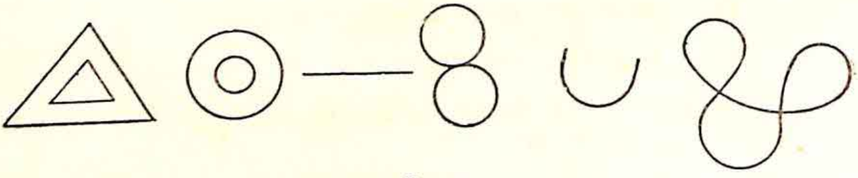
Fig. 331.



Set A

Fig. 332.

Surely we can do so. Now can we deform our triangle ABC made of an elastic band into any one of the following diagrams ?



Set B

Fig. 333.

We cannot. The same will be the case if the triangle ABC is drawn on infinitely stretchable sheet of rubber. By stretching the rubber sheet judiciously we can deform the triangle ABC into any one of the diagrams in set A but not into any one of the diagrams of of the set B.

All the figures in set A have one property in common with the original triangle and also with each other. This property is that all these figures divide the plane of the paper into parts, an inside and an outside. Each of these has one inside and one outside. In mathematical terminology these figures are said to be topologically equivalent and the geometry that discusses such properties is known as *Topology*.

Topology is that grometry which studies those properties of figures that remain unchanged when the figures are stretched or twisted in any way without being torn. As the surface of an infinitely stretchable sheet of rubber very much helps in explaining the relation of topological equivalence, Topology has also been called rubber sheet geometry.

The first figure of set B has two outsides and one inside. The second figure has also two outsides and one inside. The third figure has no inside. The fourth figure has two insides and one outside. The fifth figure has no inside. The sixth figure has three insides and one outside. So none of the figures of set B is topological equivalent to the figures of set A. The first and second figure of the set B are however topologically equivalent.

The topological properties of figures of set A that can be easily noticed are :

(1) Any of segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$ does not cross itself.

(2) If there is a point D on $\overline{AB}$ such that A—D—B then this order is preserved in any of the topologically equivalent figures.

(3) Any point that is element of an inside of any figure remains to be an element of inside of any figure that is obtained when we deform the first figure.

So, in general, the properties that topology studies are those of betweenness, order, interior, exterior etc. It will not be out of place to point out that these concepts of order, betweenness, interior and exterior were the one that were ignored by Euclid. The word 'geometry' means the 'measurement of earth' but the concept of measurement does not have any place in this new geometry nor does it involve calculations with quantities. Instead it studies the property of 'position' or relative position. Topology is also known as 'geometry of position'.

Topology had its origin in the problem of seven bridges of Konigsberg in Prussia. Konigsberg lies on Pregal river which has two islands. There was an island A called 'Kneiphof' surrounded by two branches of the river as shown in the figure. There were seven bridges *a*, *b*, *c*, *d*, *e*, *f* and *g* crossing the two bridges. The problem was that the residents of the place wanted to plan their walk in such a way that they cross each one these bridges once but not more than once. None of the residents could do so. Some had doubts about its possibility. Ultimately the problem was referred to the mathematician Euler (1707—1783) and he was asked to plan such a route whereby it may be possible to cross each of these bridges once and only once. He proved that no such route was

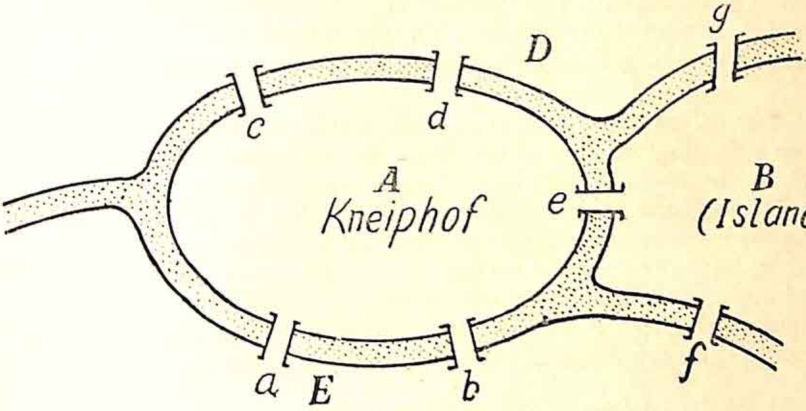


Fig. 334.

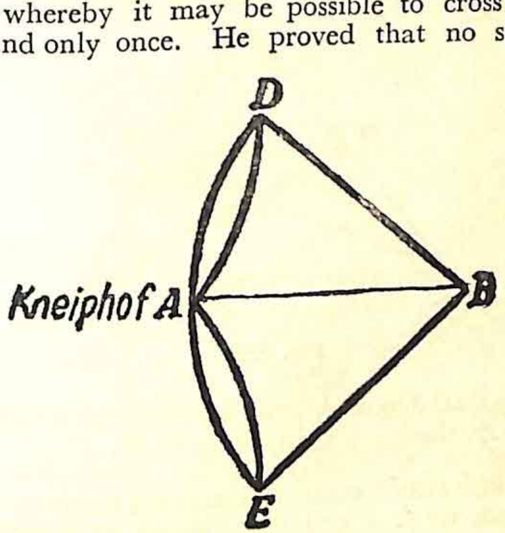


Fig. 335.

possible by reducing it to a topological problem. He contended that sizes and shapes of bridges and islands were not of any significance if we are interested in finding a solution to the problem such as mentioned above. Euler thought of land mass as points and bridges as lines and reduced the map of Kneiphof and pregal to the figure 335 on page 465.

The four points A, D, B, E represent the four land masses as shown in the map of the islands and river. The two segments connecting A and D represent the bridges *e* and *d* while the two segments connecting A and E represent the bridges *a* and *b*. The segment $\overline{AB}$ represents the bridge *e*. The segment $\overline{BD}$ represents the bridge *g* and the segment $\overline{EB}$ represents the bridge *f*. The segments connecting E and A, represent the bridges *b* and *a*.

Euler called this diagram a 'net work'. It is fairly easy to see from this diagram that if we want to traverse this network in one trip without going over any of its branches twice, we cannot do so. Had there been an even number of branches at every vertex—one leaving for every one arriving we could have done so. The exceptions can, however, be the vertex which we leave without having arrived and the vertex where we arrive at without having to leave. In other words, that at all vertices of such a network excepting two, there should be even number of branches converging.

At all the vertices of the network of koenigsberg problem, odd number of branches converge. Three of the vertices have three branches each while one vertex has five. The following network ABCDE can be traversed since all vertices excepting A and E have even number of vertices.

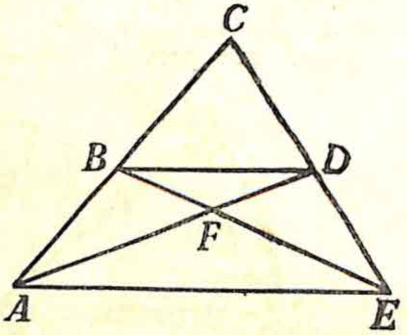


Fig. 336.

We have to start from A and end at E or the other way round. Of course, there are many ways to go.

If all the vertices of a network are even, then we can start from anywhere and then we shall end at the vertex at which we started.

An example of such a network would be :

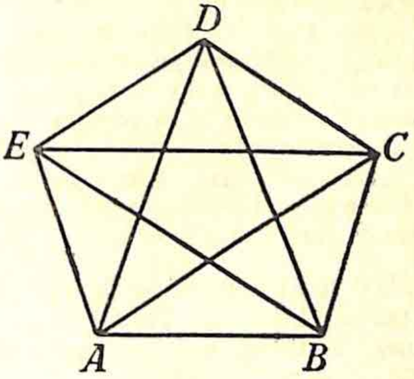
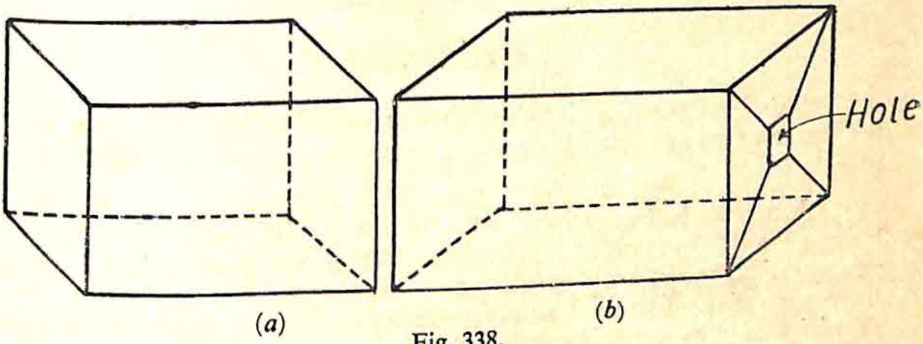


Fig. 337.

If there are four odd vertices in any network, then it can be traversed in two trips. If there are six odd vertices then it can be traversed in three trips and so on.

Topology has helped in proving a very important fact. The study of properties of polyhedral occupied a very important place in Greek Geometry but it was left to Descartes and Euler to discover a very important property of a polyhedron. By polyhedron we mean a solid consisting of a number of polygonal faces. A polyhedron is called a regular polyhedron if all the polygons are congruent and all the angles at vertices are congruent. A polyhedron will be a simple polyhedron when it does not have any 'holes' so that its surface could be deformed into a surface of a sphere. In other words, a simple polyhedron is topologically equivalent to a sphere.



(a) (b)

Fig. 338.

Fig. (a) is a simple polyhedron while fig. (b) is not as it has a 'hole'.

Let V denote the number of vertices of a simple polyhedron. Also let E be the number of edges and F the number of faces then Euler discovered that

$$V-E+F=2$$

always holds. This formula however does not hold in the case of non-simple polyhedron. Topological methods help us in proving that this formula holds in case of all simple polyhedron. It may however be furtner stated that this formula holds not only in the case of simple polyhedron with straight edges and flat faces but also in the case of simple polyhedron with curved faces and edges or any subdivision of surface of a sphere into regions bounded by curved arcs. Moreover if the surface of a sphere or a polyhedron is made up of a thin sheet of rubber, the Euler formula will hold if its surface is deformed into any other shape by stretching or bending so long as rubber is not torn in the process.

Topology deals with all sorts of configurations, surface etc. and topological properties of these like Euler formula are of greatest importance in any mathematical investigation as these properties persist under drastic changes of shape.

Many magicians' tricks are based on topological properties of knots.

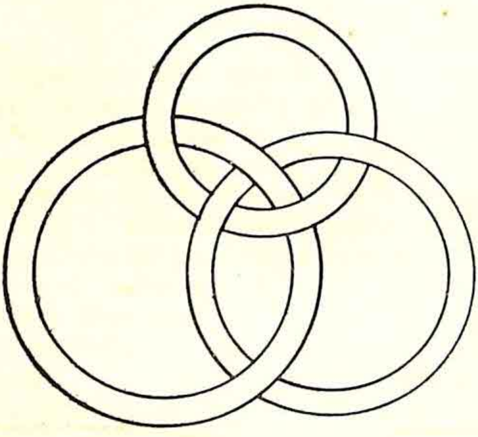


Fig. 339.

This configuration involves three rings of which all are connected but no two are not. If one ring is removed, the other two will be free.

A topological question that remains to be solved is what is known as 'Four colour problem'. When we give colours to a geographical map, we give different colours to the countries that have a portion of their boundary in common. It has been found empirically (by experiment) that any map, no matter how many countries it may contain or how these may be situated can be coloured by using only four different colours but no mathematician has so far come forward with a proof that four colours are sufficient for the purpose. However it has been proved that five colours are always enough. Now the challenge before the reader is to either draw a map that requires more than four (five say) colours or to prove mathematically that four colours are always enough.

It is however easy to see that less than four colours are not sufficient for the purpose.

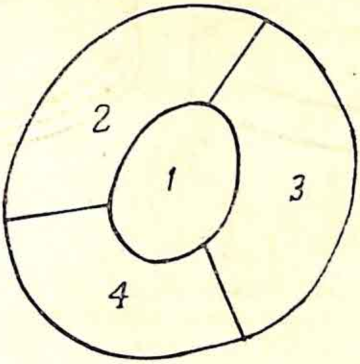


Fig. 340.

In this figure, there is a map of four countries and these countries are situated in such a manner that each of these touches the other three. So to colour such a map, three colours are not sufficient. This four colour problem can be reduced to following mathematical problem.

'If a plane is divided into any number of regions which are not overlapping than each of these regions can be marked with one of the numbers 1, 2, 3, 4 in such a way that no two adjacent regions are marked by the same number.'

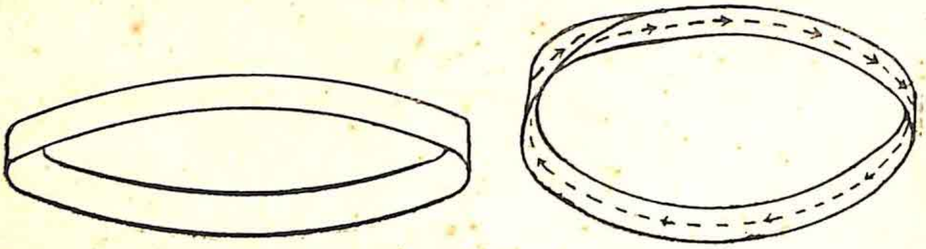
The reader is challenged to give a mathematical proof of this.

It may however be stated that 'four colour problem' will be valid even if the map is drawn on the surface of a sphere. A small hole can be bored into the sphere and then the resulting surface can be deformed into a plane. The place where hole is bored will take place of region 1 of the map given earlier.

Topology also studies those surfaces which have only 'one-side'. Can you think of such a surface ? An ordinary surface has two sides. There are two sides of surfaces of a sphere—an interior and an exterior. The two sides of the surface of a sphere could be coloured by two different colours and in this way we can distinguish the interior and the exterior. A bug crawling in the inside of such a sphere will always remain inside unless it is provided with a hole to come out.

Moebius made a surprising discovery that surfaces with only one side also exist . To explain this let us perform an experiment.

Let us have a narrow but long strip of paper and paste its ends together. We get a sort of belt. This surface has two sides. A



(*a*) (*b*)

Fig. 341.

bug crawling outside will remain outside if it is allowed to cross the edge. Similarly a bug crawling inside will remain inside.

Now suppose we give one of the ends of the strip a half twist before pasting the two ends. The surface that we get in this way can be represented by Fig. 341(*b*). Now this surface has only one edge side. A bug can crawl over all of it without crossing its edge. This surface is known as Moebius strip. Now this surface has one more interesting property. If this strip is cut all along its middle, two separate but intertwined strips are formed. It is really interesting to play with this strip by cutting it all along through lines parallel to its boundary curve and at distance $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{9}$ etc. The school students will really enjoy playing with it and Moebius strip need to be included into any elementary geometrical instruction.

Yet another interesting one sided surface is the 'Klein bottle'. It is a closed surface but does not have any inside or outside. It is a sort of a three dimensional Moebius strip.

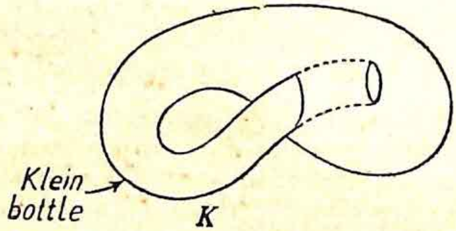


Fig. 342.

It may however be stated that topology is not a study of some tricks or isolated puzzles. Its theorems are derived through deductive reasoning and these theorems are basis of advanced work in many fields of Mathematics. For any research worker, in Mathematics, the study of topology is a 'must'.

Four dimensional geometry.

So far we have been talking about one-dimensional, two-dimensional or three-dimensional geometry. If we restrict ourselves to a line and its properties then it will be a one-dimensional geometry. If we restrict ourselves to a plane determined by two lines which are perpendicular to each other or which are inclined at any angle to each other and study its properties then it will be two dimensional geometry. Then if we study the properties of a space determined by a plane and a line perpendicular to the plane then it will be the three dimensional geometry. But what about four or more dimensions. Cannot we think of a 'hyper space' determined by a cube or parallelopiped representing space and a direction perpendicular to itself. Of course we run out of directions and physically we cannot do it. However we can build up a geometry of four dimensions by imagining the possibility of doing so.

This idea of 'fourth-dimension' has been suggested by the famous Einsteins theory of relativity. However we know that the geometry that we have so far been developing was a logical consequence of a consistent set of axioms and the idea of one dimension, two dimensions or three dimensions was not much made use of. So we can develop a geometry of four dimensions (geometry of four dimensional Euclidean space or E_4) by taking one additional undefined term and four new postulates. E_4 will also include points that do not belong to E_3 or three-dimensional Euclidean space. Hence in our new space or 'hyper-space' there will exist many three dimensional spaces just as there existed many planes (or E_2) in E_3. Hence we will have an undefined term for these various three dimensional space. A name very commonly used is 'solid', 'cube' or 'prime.'

In E_3 we had point, line and plane as our undefined terms. In E_4 we will have point, line, plane and prime as undefined terms. Four postulates will be taken in addition to six postulates of incidence that we took in E_3. The ten postulates are :

1. A line is a set of points and contains atleast two points.

2. If P, Q are any two points then there is one and only one line that contains them both.

3. Every plane is a set of points and contains atleast three non-colliner points.

4. If P, Q, R are any three non-collinear points, there is one and only one plane which contains them.

5. Every prime is a set of points and contains atleast four points which are neither collinear nor coplanar.

6. If P, Q, R, S are any four non coplanar points there is one and only one prime that contains them.

7. Space contains atleast five points which are neither collinear, coplanar, nor coprimal.

8. If two points of a line lie in a plane then every point of the line lies in the plane.

9. If three non-collinear points of a line lie in a prime, then every point of the plane lies in the prime.

10. If a plane and a prime have a point in common, their intersection is a line.

For developing E_4 geometry, other postulates that need to be taken are (*a*) Parallel postulate (*b*) Prime separation postulate (*c*) Distance measurement postulates (*d*) Angle measurement postulates (*e*) Area measurement postulates (*f*) Congruence postulate.

Let the reader develop his own E_4 geometry by making use of these postulates and let him then go on to E_5, E_6, E_7, E_8, E_9,...E_n.

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